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METABELIAN GROUPS AND VARIETIES

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CONTENTS

The principal concern of this paper is varieties of metabelian groups, and we obtain generalizations of results of Cossey (1966), Brisley (1967) and Weichsel (1967) with classifications of certain metabelian varieties of finite exponent. Examples are given to show that the lattice of metabelian varieties is not distributive. To do these things requires a discussion of varieties of certain universal algebras closely related to groups, called split-groups, which arise in a natural way from non-nilpotent metabelian critical groups. Work by Brooks (1968) and L. G. Kovács & M. F. Newman (unpublished) make it seem likely that in the near future a complete classification of all non-nilpotent, join-irreducible, metabelian varieties will be obtained. For varieties of certain split-groups closely related to metabelian groups we obtain such a description.

Chapter 0

0.1. *Introduction*

The bulk of this paper is concerned with the varietal properties of certain rather special universal algebras, called split-groups. As the name implies, a split-group is a group with a number of given splittings. The definitions and basic properties are developed in chapter 1, along the general lines of Hanna Neumann's book (1967), the aim being to provide a firm basis for the later chapters. There are many questions that one might ask about varieties of split-groups—practically all meaningful questions about varieties of groups are meaningful here—but with a few minor exceptions we ignore them as not being relevant to our purpose. This purpose could be broadly defined as the shedding of light on the nature of varieties of metabelian groups.

From the work of L. G. Kovács & M. F. Newman (unpublished), Brooks (1968) and the present work it has become clear that, while the possibility of classifying all metabelian varieties is at present slight, there is a good chance of describing all the join-irreducible ones which are not nilpotent. Kovács & Newman (see (6.1.1), (6.1.2) of the present paper) have provided a reduction of this problem to the case of finite exponent. It was in tackling an aspect of this residual case that I had the idea of split-groups; and it turns out that classification of certain join-irreducible varieties of split-groups is necessary to the classification of join-irreducible varieties of metabelian groups. Chapters 3 and 4, though couched in split-group language, are of direct relevance to this problem, and reduce the problem further to the case of prime-power exponent.

Chapters 4 and 5 have an indirect relevance to varieties of metabelian groups in that, for the variety of split-groups $\mathfrak{A} \circ \mathfrak{A}$ (the class of all split-groups with a prescribed abelian-by-abelian splitting), we describe all non-nilpotent join-irreducible subvarieties. For varieties of metabelian groups this information is still incomplete, though recently Brooks (1968) has determined all non-nilpotent join-irreducible subvarieties of $\mathfrak{A}_p\,\mathfrak{A}_{p^2}$ (p prime), and there is hope that his methods will generalize to the case of arbitrary prime-power exponent (perhaps along the lines of § 4.2). Chapter 5 depends heavily on work of L. G. Kovács & M. F. Newman (unpublished) and I thank them for allowing me to adapt their methods. Proofs of the relevant results of theirs ((6.1.1), (6.1.2)) are sketched in appendix I; a fuller acknowledgement is given in the introduction to chapter 5.

Chapter 6 is application to varieties of metabelian groups. We derive a complete description of the lattice of subvarieties of certain metabelian varieties $\mathfrak{B}$ in the following cases; first when $\mathfrak{B}$ is $\mathfrak{A}_m\,\mathfrak{A}_n$ with m nearly prime to n ($p \mid m$ implies $p^2 \nmid n$), thus generalizing work of Cossey (1966) who deals with the coprime case, and of Kovács & Newman (unpublished) who handle the case $m = p^\alpha$, $n = p$; and second when $\mathfrak{B}$ has finite exponent and p-groups in $\mathfrak{B}$ have class at most p, thus generalizing Weichsel (1967) and Brisley (1967). We also have results about the distributivity of the lattice of subvarieties of arbitrary $\mathfrak{A}_m\,\mathfrak{A}_n$.

L. G. Kovács has pointed out that split-groups of species 2 can be thought of as group pairs, in the terminology of Plotkin (1966). Chapters 1 to 5 here therefore contain, so far as I am aware, the first detailed results on specific varieties of group pairs.

A résumé of this paper is contained in Bryce (1969).

0.2. *Notation and terminology*

For results relating to varieties of universal algebras we refer the reader to B. H. Neumann (1962), and, for results and notation relating specifically to varieties of groups, to Hanna Neumann (1967).

We differ from the latter only in writing $H \leqslant G$ if H is a subgroup of G. If H is a proper subgroup of G, that is $H \neq G$, we write $H < G$. If H is normal in G we write $H \trianglelefteq G$. Write $G = \langle H_1, \ldots, H_r \rangle$ if G is generated by the subsets $H_1, \ldots, H_r$.

If G is a group and x, y are elements of G denote $y^{-1}xy$ by x^y and the commutator $x^{-1}x^y$ by $[x, y]$. Commutators of higher weight are defined as left-normed: if $x_1, \ldots, x_n$ belong to G and $[x_1, \ldots, x_{n-1}]$ has been defined, then

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

Define $[x, 0y] = x$, and for $r \geqslant 0$, $[x, (r+1)\,y] = [[x, ry], y]$.

If H, K are subsets of G, then $[H, K]$ is the subgroup of G generated by the elements $[h, k]$ $(h \in H,\ k \in K)$. The derived group $G' = G_{(2)}$ of G is $[G, G]$. A group G is metabelian if $[G', G'] = 1$, where we use 1 to denote the identity of the group as well as the trivial subgroup. The normal closure of H in G is denoted by H^G.

The lower central series of a group G is defined as in Hanna Neumann (1967, 12.82). The centralizer of a subgroup H of G is denoted by $C_G(H)$, and the centre of G by $Z(G)$. The Fitting subgroup of a finite group G, the largest normal nilpotent subgroup of G, is denoted by $F(G)$.

A finite group with a unique minimal normal subgroup is called *monolithic*, and the unique minimal normal subgroup is the *monolith*. The *socle* of a finite group G is the subgroup generated by all minimal normal subgroups of G and is denoted by σG.

In later chapters, Chapter 4 in particular, many well-known commutator identities will be used, often without comment. The ones used are listed here. In any group G the following are identities:

$$[x, yz] = [x, z]\,[x, y]\,[x, y, z],$$
$$[xy, z] = [x, z]\,[x, z, y]\,[y, z],$$
$$[x, y] = [y, x]^{-1},$$
$$[x, y^{-1}] = [x, y]^{-y^{-1}}.$$

In a metabelian group G:

$$[x, y, z]\,[y, z, x]\,[z, x, y] = 1;$$

therefore, if d is in $C_G(G')$,

$$[d, x, y] = [d, y, x];$$

for all natural numbers r,

$$[x, y^r] = \prod_{i=1}^{r} [x, iy]^{\binom{r}{i}}, \tag{0.2.1}$$

$$[x, ryz] = \prod_{i=0}^{r} \prod_{j=r-i}^{r} [x, iy, jz]^{\binom{r}{i}\binom{i}{i+j-r}}. \tag{0.2.2}$$

I^+ will be used to denote the set of natural numbers.

Chapter 1. Varieties of split-groups

In this chapter we are concerned with varieties of certain objects called split-groups, which are defined below. A split-group is, suitably interpreted, a universal algebra, and this is pointed out in § 1.2; hence much general theory is applicable to our situation, and it will be called on to eliminate long proofs which would be redundant. However, our interest in varieties of split-groups, or split-varieties for short, is the way they can be used to give results about varieties of groups; more insight seems to be gained by developing the theory of split-groups as is done below, than is gained by regarding split-groups and varieties of split-groups as part of a much more general framework.

1.1. *Split-groups*

Definition. *A split-group of the species n, is an* $(n+1)$*-tuple* $(G, A_1, \ldots, A_n)$ *where G is a group,* $A_1, \ldots, A_n$ *are subgroups generating G such that, if* $B_i = \langle A_i, \ldots, A_n \rangle$, $i \in \{1, \ldots, n\}$, *then* A_i *is normal in* B_i *and is complemented in* B_i *by* B_{i+1}:

$$A_i \trianglelefteq B_i, \quad A_i B_{i+1} = B_i, \quad A_i \cap B_{i+1} = 1.$$

We shall denote the split-group $(G, A_1, \ldots, A_n)$ *by* $\mathbf{G}$ *when no confusion can arise as to the particular splitting of G involved; also we may write* $A_i = A_i(\mathbf{G})$, $B_i = B_i(\mathbf{G})$, $i \in \{1, \ldots, n\}$. *The group G is called the* carrier *of* $\mathbf{G}$; *an* element *of* $\mathbf{G}$ *is an element of G.* (1.1.1)

Definition. *A* sub-split-group *of the split-group* $(G, A_1, \ldots, A_n)$ *is a split-group* $(G^*, A_1^*, \ldots, A_n^*)$ *where* G^* *is a subgroup of G and where* $A_i^* = A_i \cap G^*$, $i \in \{1, \ldots, n\}$. *A sub-split-group is* normal *if it is normal as a subgroup.* (1.1.2)

Definition. *A* morphism μ *between two split-groups* $(G, A_1, \ldots, A_n)$ *and* $(G^*, A_1^*, \ldots, A_n^*)$ *is a group homomorphism* $\mu: G \to G^*$ *such that* $A_i \mu \leqslant A_i^*$, $i \in \{1, \ldots, n\}$. *We write* $\mu: \mathbf{G} \to \mathbf{G}^*$. (1.1.3)

Notice that morphisms are defined only between split-groups of the same species; this dependence on the species will often be left understood, unless it is necessary to clarify the meaning. Note also that, in general, an inner automorphism of G is not a self-morphism of $\mathbf{G}$.

Definition. *A morphism is* epi *or* mono *according as it is onto or one-to-one as a group homomorphism of the carriers.* (1.1.4)

Definition. *If* $\mathbf{G} = (G, A_1, \ldots, A_n)$ *is a split-group and* $\mathbf{N}$ *is a normal sub-split-group of* $\mathbf{G}$, *the quotient split-group* $\mathbf{G}/\mathbf{N}$ *is the split group*

$$\mathbf{G}/\mathbf{N} = (G/N, A_1 N/N, \ldots, A_n N/N). \tag{1.1.5}$$

The right-hand side is indeed a split-group: clearly $A_i N/N \trianglelefteq B_i N/N$ and if $a_i \in A_i$, $b_{i+1} \in B_{i+1}$ such that $a_i N = b_{i+1} N$ then $b_{i+1}^{-1} a_i \in N = (N \cap A_1) \ldots (N \cap A_n)$ which implies $a_i \in N$ by the uniqueness of the decomposition $g = a_1 a_2 \ldots a_n$ for an element g of G.

Lemma *If* $\mu: \mathbf{G} \to \mathbf{G}^*$ *is a morphism between two split-groups then*

$$(\ker \mu, \ker \mu | A_1(\mathbf{G}), \ldots, \ker \mu | A_n(\mathbf{G}))$$

is a normal sub-split-group of $\mathbf{G}$. *(Here* $\mu | A_i(\mathbf{G})$ *denotes the restriction of* μ *to* $A_i(\mathbf{G})$.) (1.1.6)

Proof. We have only to verify that $\ker \mu$ splits appropriately; indeed if $a_1 a_2 \ldots a_n \in \ker \mu$ with $a_i \in A_i(\mathbf{G})$, then $(a_1 \mu)(a_2 \mu) \ldots (a_n \mu) = 1$ so that $a_1 \mu = \ldots = a_n \mu = 1$, or $a_i \in \ker \mu | A_i(\mathbf{G})$, $i \in \{1, \ldots, n\}$.

Definition. *The* cartesian product *of a collection of split-groups* $\mathbf{G}_i = (G_i, A_{i1}, \ldots, A_{in})$ $(i \in I)$ *of the same species is the split-group* $\mathbf{G}$ *where* $G = \Pi\{G_i : i \in I\}$ *and where* $A_j(\mathbf{G}) = \Pi\{A_{ij} : i \in I\}$ *is embedded in* G *in the natural way*:

$$A_j(\mathbf{G}) = \{f \in G : f(i) \in A_{ij},\ i \in I\}.$$

The restricted direct product *is defined similarly.* (1.1.7)

Definition. *A* fully invariant *sub-split-group of* $\mathbf{G}$ *is one invariant under all self-morphisms of* $\mathbf{G}$. (1.1.8)

Note that, as not every inner automorphism of G is a self-morphism of $\mathbf{G}$, a fully invariant sub-split-group need not be normal; for example $A_i(\mathbf{G})$ is fully invariant, but of course not necessarily normal, in $\mathbf{G}$. It is easy to see that the intersection of the normal sub-split-groups which contain a given fully invariant sub-split-group is fully invariant (and normal).

Definition. *A generating set* $\{a_{ij} \in A_i(\mathbf{G}) : j \in J_i,\ 1 \leqslant i \leqslant n\}$ *of* G *will be called a* generating set *of* $\mathbf{G}$. *A split-group is* finitely generated *if it has a finite generating set.* (1.1.9)

A split-group will be said to have a certain property if its carrier has the property; thus $\mathbf{G}$ is *finite* if G is finite. For split-groups of small species, special names will be adopted: a split-group of species 2 is a *bigroup*, and one of species 3 is a *trigroup*.

Finally, in this section, we note a few abuses of language that will occur from time to time. The trivial split-group should, of course, be written as $\mathbf{1} = (1, 1, \ldots, 1)$ but we will write 1 for it, and also for the trivial sub-split-group of a split group. A subgroup S of G may be referred to as 'the sub-split-group S' of G if it splits appropriately, while a sub-split-group may be referred to as a subgroup if, by doing so, the desired emphasis is conveyed without creating confusion.

1.2. *Alternative formulation*

We shall in this section characterize split-groups as certain universal algebras. We repeat that our reference for results on universal algebras is Neumann (1962). The operator domain is defined as follows.

Definition. Ω_n *is a commutative semigroup* $\{\omega_0, \omega_1, \ldots, \omega_n\}$ *of order* $n+1$ *with multiplication table*

$$\omega_i\,\omega_j = \omega_j \quad \textit{for} \quad 0 \leqslant i \leqslant j \leqslant n. \tag{1.2.1}$$

In the terminology of Clifford & Preston (1961), Ω_n is a *commutative band*, fully ordered with respect to the relation: $\omega_i \leqslant \omega_j$ if and only if $\omega_i\,\omega_j = \omega_j$.

Definition. *An* Ω_n*-group is a triple* $(G, \Omega_n, \mathrm{e})$, *where* G *is a group and where the mapping* $\mathrm{e} : G \times \Omega_n \to G$ *has the properties*

$$(xy)\,\omega_i\,\mathrm{e} = (x\omega_i\,\mathrm{e})\,(y\omega_i\,\mathrm{e}),$$

$$x\omega_0\,\mathrm{e} = x, \quad x\omega_n\,\mathrm{e} = 1,$$

and $$(x\omega_i\,\mathrm{e})\,\omega_j\,\mathrm{e} = x(\omega_i\,\omega_j)\,\mathrm{e},$$

for all $x, y \in G$ and $i, j \in \{0, 1, \ldots, n\}$. (1.2.2)

Since an Ω_n-group is a universal algebra, the concepts of sub-Ω_n-group, quotient Ω_n-group have standard definitions; we give them here using the well-known correspondence between congruences on groups and normal subgroups.

Definition. *A sub-*Ω_n*-group of an* Ω_n*-group* $(G, \Omega_n, \mathrm{e})$ *is an* Ω_n*-group* $(G^*, \Omega_n, \mathrm{e}^*)$ *where* G^* *is a subgroup of* G *and where* $\mathrm{e}^* = \mathrm{e}|G^* \times \Omega_n$. (1.2.3)

Definition. *If* $(G, \Omega_n, \mathfrak{e})$ *is an* Ω_n*-group and* $(N, \Omega_n, \mathfrak{e}')$ *is a normal sub-*Ω_n*-group (that is, a sub-*Ω_n*-group which is normal qua subgroup), then the quotient* Ω_n*-group* $(G, \Omega_n, \mathfrak{e})/(N, \Omega_n, \mathfrak{e}')$ *is the* Ω_n*-group* $(G/N, \Omega_n, \mathfrak{e}'')$ *where* $\mathfrak{e}''$: $G/N \times \Omega_n \to G/N$ *is defined by* $xN\omega_i \mathfrak{e}'' = x\omega_i \mathfrak{e} N$. (1.2.4)

Definition. *A homomorphism* μ: $(G, \Omega_n, \mathfrak{e}) \to (G^*, \Omega_n, \mathfrak{e}^*)$ *between* Ω_n*-groups is a group homomorphism* μ: $G \to G^*$ *such that for all* $x \in G$, $(x\omega_i \mathfrak{e})\mu = (x\mu)\,\omega_i \mathfrak{e}^*$. (1.2.5)

Definition. *The cartesian product of a collection* $(G_i, \Omega_n, \mathfrak{e}_i)$ $(i \in I)$ *of* Ω_n*-groups is the* Ω_n*-group* $(G, \Omega_n, \mathfrak{e})$, *where* $G = \Pi\{G_i : i \in I\}$ *and where* $\mathfrak{e}$: $G \times \Omega_n \to G$ *is defined by* $f\omega_j \mathfrak{e}(i) = f(i)\,\omega_j \mathfrak{e}_i$, $f \in G$, $i \in I, j \in \{0, \ldots, n\}$. (1.2.6)

Theorem. *There is a functor* Φ *from the category of all split-groups of species n to the category of all* Ω_n*-groups, which is one-to-one on both objects and morphisms and which preserves sub-structures, quotient structures and cartesian products.* (1.2.7)

Proof. Let $(G, A_1, \ldots, A_n)$ be a split-group. Define the endomorphisms σ_i of G by

$$(a_1 a_2 \ldots a_n)\,\sigma_i = a_{i+1} \ldots a_n$$

for all $a_j \in A_j$, $j \in \{1, \ldots, n\}$, $i \in \{0, 1, \ldots, n-1\}$; and define σ_n to be the zero endomorphism of G. We call σ_i the *splitting endomorphisms* of $\boldsymbol{G}$. Clearly

$$\left.\begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i = \sigma_j, \quad 0 \leqslant i \leqslant j \leqslant n, \\ \sigma_0 = 1_G, \quad \sigma_n = 0_G. \end{array}\right\} \tag{1.2.8}$$

Also $B_{i+1} = G\sigma_i$ and $A_i = \ker \sigma_i \cap B_i$, $i \in \{0, \ldots, n\}$. Conversely, if a group G has endomorphisms σ_i with the properties (1.2.8), then by writing $B_{i+1} = G\sigma_i$, $A_i = \ker \sigma_i \cap B_i$, $i \in \{0, \ldots, n\}$, $(G, A_1, \ldots, A_n)$ is a split-group. For, if $x \in B_{i+1}$ then

$$x = x\sigma_i = ((x\sigma_i)\,(x\sigma_{i+1})^{-1})\,(x\sigma_{i+1})$$

and $((x\sigma_i)\,(x\sigma_{i+1})^{-1})\,\sigma_{i+1} = (x\sigma_{i+1})\,(x\sigma_{i+1})^{-1} = 1$, so that $(x\sigma_i)\,(x\sigma_{i+1})^{-1} \in \ker \sigma_{i+1} \cap B_{i+1} = A_{i+1}$, which shows that $B_{i+1} = A_{i+1} B_{i+2}$. Also $A_{i+1} \trianglelefteq B_{i+1}$, and if $y \in A_{i+1} \cap B_{i+2}$ then there exists $y_1 \in G$ with $y = y_1 \sigma_{i+1}$, whence

$$1 = y\sigma_{i+1} = y_1 \sigma_{i+1}^2 = y_1 \sigma_{i+1} = y$$

and therefore $A_{i+1} \cap B_{i+2} = 1$. This shows that $(G, A_1, \ldots, A_n)$ is a split-group.

If $\boldsymbol{G} = (G, A_1, \ldots, A_n)$ is a split-group, define $\boldsymbol{G}\Phi = (G, \Omega_n, \mathfrak{e})$ where $\mathfrak{e} : G \times \Omega_n \to G$ is given by

$$x\omega_i \mathfrak{e} = x\sigma_i, \quad i \in \{0, \ldots, n\}, \tag{1.2.9}$$

for all $x \in G$. Conversely, if $(G, \Omega_n, \mathfrak{e})$ is an Ω_n-group we use (1.2.9) to define endomorphisms σ_i of G, which may easily be verified to have the properties (1.2.8), and therefore, in this way, $(G, \Omega_n, \mathfrak{e})$ defines a unique split-group $(G, \Omega_n\, \mathfrak{e})\,\Psi$. Clearly $\Phi\Psi$ is the identity mapping on the class of all split-groups of species n, and $\Psi\Phi$ is the identity mapping in the class of all Ω_n-groups, hence Φ is one-to-one and onto on objects.

If $\mu : \boldsymbol{G} \to \boldsymbol{G}^*$ is a morphism, then $\mu : \boldsymbol{G}\Phi \to \boldsymbol{G}^*\Phi$ is a homomorphism; for it is easy to verify that if σ_i, σ_i^* are the splitting endomorphisms corresponding to $\boldsymbol{G}$, $\boldsymbol{G}^*$ respectively, then $\sigma_i \mu = \mu\sigma_i^*$, $i = \{0, \ldots, n\}$. Hence, from (1.2.9)

$$(x\omega_i \mathfrak{e})\,\mu = x\sigma_i \mu = x\mu\sigma_i^* = (x\mu)\,\omega_i \mathfrak{e}^*$$

for all $x \in G$. Conversely, every $\mu : \boldsymbol{G}\Phi \to \boldsymbol{G}^*\Phi$ is a morphism $\mu : \boldsymbol{G} \to \boldsymbol{G}^*$. If we put $\mu\Phi = \mu$, then clearly Φ is a functor. The rest of the theorem is proved by similar techniques which we omit.

We may use definition 1.2.2 to appeal to general results: for example, the usual homomorphism theorems apply for Ω_n-groups, and therefore, via theorem 1.2.7, for split-groups also. Because of the application we wish to make, and for convenience in simplifying notation in the calculations of chapter 4, it is the split-group definition rather than the Ω_n-group definition that we use. In the sequel we shall suppress statements in the Ω_n-group formulation except if the comparison is of interest (for example we are led to different definitions of free objects), or if brevity can be obtained by appeal to more general results.

1.3. *Freeness of split-groups*

Let $Y_1, \ldots, Y_n$ be free groups of rank $m_1, \ldots, m_n$ respectively, on the free generating sets $\{y_{ij}: j \in J_i\}$ ($|J_i| = m_i$); we do not insist that the m_i be finite cardinals. Put $\boldsymbol{m} = (m_1, \ldots, m_n)$ and define the split group $\boldsymbol{Q}(\boldsymbol{m})$ as follows: the carrier is the free product $Y_1 * \ldots * Y_n$, and

$$A_i(\boldsymbol{Q}(\boldsymbol{m})) = Y_i^{\langle Y_i, \ldots, Y_n \rangle} \tag{1.3.1}$$

the normal closure of Y_i in $\langle Y_i, \ldots, Y_n \rangle$. As a matter of convenience we follow Hanna Neumann (1967) and use ∞ for the cardinal of the natural numbers; additionally we write $\boldsymbol{\infty}$ for the n-tuple $(\infty, \ldots, \infty)$.

Definition. $\boldsymbol{Q}(\boldsymbol{m})$ *is the* absolutely split-free split-group of rank $\boldsymbol{m}$ *on the split-free generating set* $\{y_{ij}: j \in J_i,\ 1 \leqslant i \leqslant n\}$. $\boldsymbol{Q}(\boldsymbol{\infty})$ *of species n we denote by* $\boldsymbol{Q}_n$. *For cardinals* $m_1, \ldots, m_n$ *no larger than* ∞, *we shall assume* $\boldsymbol{Q}(\boldsymbol{m})$ *embedded in* $\boldsymbol{Q}_n$ *in the natural way.*

The use of the word 'rank' needs justifying and we will cover this in (1.3.6). Where the context allows, the inelegant prefix 'split' will be dropped.

Theorem. *If* $\boldsymbol{G}$ *is a split-group of species n then every set of mappings* $\mu_i: \{y_{ij}: j \in J_i\} \to A_i(\boldsymbol{G})$ *can be extended to a morphism* $\mu: \boldsymbol{Q}(m_1, \ldots, m_n) \to \boldsymbol{G}$. (1.3.2)

Proof. Since $Q(m_1, \ldots, m_n)$ is a free group with the y_{ij}'s as a free generating set, certainly a group homomorphism μ, which extends all μ_i, exists; that $A_i(\boldsymbol{Q}(m_1, \ldots, m_n))\,\mu$ is contained in A_i follows from the definition of $A_i(\boldsymbol{Q}(m_1, \ldots, m_n))$ and the fact that $A_i(\boldsymbol{G})$ is normal in $\langle A_i(\boldsymbol{G}), \ldots, A_n(\boldsymbol{G})\rangle$.

As in more general situations, we have the concept of relative freeness, and theorems characterizing it.

Definition. *A split-group* $\boldsymbol{G}$ *of species n is* relatively split-free *if it has a generating set*

$$\{a_{ij}: j \in J_i,\ 1 \leqslant i \leqslant n\}$$

with $1 \neq a_{ij} \in A_i(\boldsymbol{G})$ *such that every set of mappings* $\mu_i: \{a_{ij}: j \in J_i\} \to A_i(\boldsymbol{G})$ *can be extended to a morphism of* $\boldsymbol{G}$ *into* $\boldsymbol{G}$. *Such a generating set is called a* split-free generating set *for* $\boldsymbol{G}$. *If* $m_i = |J_i|$, $(m_1, \ldots, m_n)$ *is called the* rank *of* $\boldsymbol{G}$. (1.3.3)

Note that in this definition, some of the m_i may be zero: this would occur if $A_i(\boldsymbol{G}) = 1$. Invariance of the rank will be proved in lemma 1.3.6.

Theorem. *If* $\boldsymbol{G}$ *is relatively split-free, then* $\boldsymbol{G}$ *has a representation* $\boldsymbol{Q}/\boldsymbol{S}$, *where* $\boldsymbol{Q}$ *is absolutely free of the same rank as* $\boldsymbol{G}$, *and* $\boldsymbol{S}$ *is a normal, fully invariant sub-split-group of* $\boldsymbol{Q}$. *Conversely, every such quotient split-group* $\boldsymbol{Q}/\boldsymbol{S}$ *is relatively split-free; if the rank of* $\boldsymbol{Q}$ *is* $(m_1, \ldots, m_n)$, *then that of* $\boldsymbol{Q}/\boldsymbol{S}$ *is* $(m'_1, \ldots, m'_n)$ *where* $m'_i = m_i$, *unless S contains* $A_i(\boldsymbol{Q})$ *in which case* $m'_i = 0$. (1.3.4)

Proof. Suppose that $\boldsymbol{G}$ is relatively split-free on the split-free generating set

$$\{a_{ij}: j \in J_i,\ 1 \leqslant i \leqslant n\}.$$

Let $Q = Q(m_1, \ldots, m_n)$ where $m_i = |J_i|$, $i \in \{1, \ldots, n\}$. Define the epimorphism $\lambda: Q \to G$ by

$$y_{ij}\lambda = a_{ij}, \quad j \in J_i,\ i \in \{1, \ldots, n\},$$

and theorem 1.3.2. Put $S = \ker \lambda$; then S carries a normal sub-split-group of Q by (1.1.6). To show that S is fully invariant, let α be an arbitrary self-morphism of Q and define the mapping $\beta: \{a_{ij} : j \in J_i,\ 1 \leqslant i \leqslant n\} \to G$ by

$$a_{ij}\beta = (y_{ij}\alpha)\lambda.$$

By definition, β can be extended to a self-morphism of G. Since the restrictions of $\alpha\lambda$ and $\lambda\beta$ to the set $\{y_{ij} : j \in J_i,\ 1 \leqslant i \leqslant n\}$ of generators of Q agree, $\alpha\lambda = \lambda\beta$. Hence if $s \in S$, $s\alpha\lambda = s\lambda\beta = 1$, and so $s\alpha \in \ker \lambda = S$.

In order to prove the converse, we need the following lemma, which was proposed to me by L. G. Kovács.

LEMMA. *Let H be relatively split-free on the generating set $h = \{h_{ij} : j \in J_i,\ 1 \leqslant i \leqslant n\}$. Let $\alpha: H \to K$ be an epimorphism such that $A_i(H) \neq 1$ implies $A_i(K) \neq 1$, and such that* $\ker\alpha$ *is fully invariant. Then $\alpha|h$ is one-to-one, and K is relatively split-free on $h\alpha$.* (1.3.5)

Proof. First, $\alpha|h$ is one-to-one. For, if $h_{ij}\alpha = h_{il}\alpha$, $j \neq l$, then $h_{ij} = h_{il}x$, $x \in \ker\alpha$. Define $\eta: H \to H$ so that $h_{ij}\eta = h_{ij}$, $h_{il}\eta = 1$; then $h_{ij} = h_{ij}\eta = x\eta \in \ker\alpha$ since $\ker\alpha$ is fully invariant. Hence $A_i(H)$ is contained in $\ker\alpha$ and therefore $A_i(K) = 1$. It follows that $\alpha|h$ is one-to-one.

Secondly, H is split-free on $h\alpha$. For let $\beta: h\alpha \to K$ by any map such that $h_{ij}\alpha\beta \in A_i(K)$. Define $\eta: H \to H$ so that $h_{ij}\eta \in h_{ij}\alpha\beta\alpha^{-1}$. Consider the map $\alpha^{-1}\eta\alpha$ from K to the set of non-empty subsets of K. Observe that $1\alpha^{-1}\eta\alpha = (\ker\alpha)\eta\alpha \leqslant (\ker\alpha)\alpha = 1$; that is $1\alpha^{-1}\eta\alpha = \{1\}$. Also, if $k = k_1^{-1}k_2$ then $k\alpha^{-1} = (k_1^{-1}\alpha^{-1}).(k_2\alpha^{-1})$ in the usual multiplication of subsets of a group, and therefore

$$k\alpha^{-1}\eta\alpha = (k_1^{-1}\alpha^{-1}\eta\alpha).(k_2\alpha^{-1}\eta\alpha).$$

Thus $\{1\} = (k^{-1}\alpha^{-1}\eta\alpha).(k\alpha^{-1}\eta\alpha)$ for all $k \in K$ showing that $|k\alpha^{-1}\eta\alpha| = 1$. Hence $\alpha^{-1}\eta\alpha$ is an endomorphism of K, and since it agrees on $h\alpha$ with β, it is a morphism $K \to K$.

We return to the proof of (1.3.4). Write Q^* for the absolutely split-free split-group of rank $(m_1', \ldots, m_n')$, where $m_i' = m_i$ unless S contains $A_i(Q)$, in which case $m_i' = 0$. Then there exists a natural morphism $\gamma: Q \to Q^*$ such that $\ker\gamma = \langle Y_i : A_i(Q) \leqslant S\rangle^Q$. If $\delta: Q \to Q/S$ is the natural morphism, define $\alpha: Q^* \to Q/S$ by

$$y_{ij}^*\alpha = y_{ij}\delta,$$

where y_{ij}^*, y_{ij} are split-free generators of Q^* and Q respectively. Clearly

$$\gamma\alpha = \delta.$$

Now $\ker\alpha$ is fully invariant in Q^*; for, if $\xi: Q^* \to Q^*$ then there exists $\eta: Q \to Q$ such that $\gamma\xi = \eta\gamma$; and if $q^* \in \ker\alpha$, there exists $q \in Q$ with $q\gamma = q^*$. Now $q\gamma\alpha = q^*\alpha = 1 = q\delta$ which means $q \in S$. Therefore

$$q^*\xi\alpha = (q\gamma\xi)\alpha = (q\eta\gamma)\alpha = (q\eta)\delta = 1$$

since $q\eta \in S$. That is, $q^*\xi \in \ker\alpha$, and therefore $\ker\alpha$ is fully invariant. Also $A_i(Q^*)$ non-trivial implies $A_i(Q/S)$ non-trivial and so the conditions of the lemma 1.3.5 are satisfied, and Q/S has the asserted properties.

LEMMA. *The rank is an invariant of a relatively split-free split-group.* (1.3.6)

Proof. Let G be relatively split-free. If $A_i(G)$ is non-trivial, then G' does not contain $A_i(G)$. For, if a_{ij} is an element of a split-free generating set, consider the self-morphism $\mu: G \to G$ such that $a_{ij}\mu = a_{ij}$ with all other split-free generators mapped to 1. Clearly G' is contained in $\ker\mu$ and a_{ij} is not in $\ker\mu$. Now G' carries a fully invariant sub-split-group of $\boldsymbol{G}$ and the hypotheses of (1.3.5) are satisfied by the natural morphism $\alpha: \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{G}'$. Hence $\boldsymbol{G}/\boldsymbol{G}'$ is relatively split-free of the same rank as $\boldsymbol{G}$; and since each $A_i(\boldsymbol{G}/\boldsymbol{G}')$ is a relatively free abelian *group*, its rank is invariant, and therefore so is that of $\boldsymbol{G}$.

To conclude this section I mention that had one treated a split-group as an Ω_n-group as discussed in § 1.2, one would have been led to a smaller class of free split-groups; indeed we can make a distinction between 'free split-group' and 'split-free split-group' as indicated by the following theorem.

Theorem. *Let* $(G, \Omega_n, \mathfrak{e})$ *be a free* Ω_n*-group in the variety of all* Ω_n*-groups, say one of rank k. Then* $(G, \Omega_n, \mathfrak{e})\,\Phi^{-1}$ *is an absolutely split-free split-group of species n, and rank* $(k, k, \ldots, k)$. (1.3.7)

Proof. Write $\boldsymbol{G} = (G, \Omega_n, \mathfrak{e})\,\Phi^{-1}$. Let $\{x_j : j \in J\}$ be a free generating set for $(G, \Omega_n, \mathfrak{e})$, $|J| = k$. Put

$$z_{ij} = (x_j \omega_{i-1}\mathfrak{e})\,(x_j \omega_i \mathfrak{e})^{-1}, \quad i \in \{1, \ldots, n\}, j \in J.$$

Then, for each $j \in J$,

$$x_j = z_{1j} z_{2j} \ldots z_{nj}.$$

It is clear, therefore, that $\{z_{ij} : j \in J_i, 1 \leqslant i \leqslant n\}$ is a generating set for $\boldsymbol{G}$. Let $\boldsymbol{H}$ be an arbitrary split-group, $\boldsymbol{H}\Phi = (H, \Omega_n, \mathfrak{e}^*)$, and $\mu_i : \{z_{ij} : j \in J_i\} \to A_i(\boldsymbol{H})$ a set of mappings. Define $\mu : \{x_j : j \in J\} \to \boldsymbol{H}\Phi$ by

$$x_j \mu = (z_{1j}\mu_1)\,(z_{2j}\mu_2) \ldots (z_{nj}\mu_n), \quad j \in J.$$

It follows that μ can be extended to a homomorphism μ: $\boldsymbol{G}\Phi \to \boldsymbol{H}\Phi$, and hence, by theorem 1.2.7, that μ: $\boldsymbol{G} \to \boldsymbol{H}$ is a morphism. It is easy to verify that μ does extend the μ_i:

$$\begin{aligned} z_{ij}\mu &= ((x_j \omega_{i-1}\mathfrak{e})\,(x_j \omega_i \mathfrak{e})^{-1})\,\mu \\ &= ((x_j\mu)\,\omega_{i-1}\mathfrak{e}^*)\,((x_j\mu)\,\omega_i \mathfrak{e}^*)^{-1} \\ &= (z_{ij}\mu_i) \ldots (z_{nj}\mu_n)\,.\,((z_{i+1j}\mu_{i+1}) \ldots (z_{nj}\mu_n))^{-1} \\ &= z_{ij}\mu_i. \end{aligned}$$

If we choose for $\boldsymbol{H}$ the split-free split-group of species n, $\boldsymbol{Q}(k, \ldots, k)$, define μ as above from $\mu_i : z_{ij} \to y_{ij}$, and $\nu : \boldsymbol{H} \to \boldsymbol{G}$ by $\nu : y_{ij} \to z_{ij}$ and theorem 1.3.2, we get that $\mu\nu = 1_G$ and $\nu\mu = 1_H$, so $\boldsymbol{G} \cong \boldsymbol{H} = \boldsymbol{Q}(k, \ldots, k)$.

1.4. *Split-words*

Definition. *A* split-word *of species n is an element of* $\boldsymbol{Q}_n$. (1.4.1)

Definition. *Two sets* S_1, S_2 *of split-words of species n are* super-equivalent *if they have the same fully invariant closure in* $\boldsymbol{Q}_n$. (1.4.2)

We shall need a version of theorem 33.45 of Hanna Neumann (1967); our adaption is made from unpublished results of L. G. Kovács & M. F. Newman which strengthen 33.45. Note that the carrier of $\boldsymbol{Q}_n$ is a free group of countably infinite rank on the free generating set

$$\{y_{ij} : j \in I^+, 1 \leqslant i \leqslant n\}.$$

Definition. *For* $i \in \{1, 2, \ldots, n\}$, $j \in I^+$, *define the self-morphisms* δ_{ij} *(called* deletions*) of* $\boldsymbol{Q}_n$ *by*

$$y_{kl}\delta_{ij} = \begin{cases} y_{kl}, & (k, l) \neq (i, j), \\ 1, & (k, l) = (i, j). \end{cases} \tag{1.4.3}$$

DEFINITION. *An element q of* $\boldsymbol{Q}_n$ *is* uniform *if for each deletion* δ_{ij}, *either* $q\delta_{ij} = 1$ *or* $q\delta_{ij} = q$. (1.4.4)

THEOREM. *If q belongs to* $\boldsymbol{Q}(m_1, \ldots, m_n)$ *for finite* m_i, *then q is a product of split-words* q_J, $J \subseteq J_1 \cup \ldots \cup J_n$ *(the* J_i *are assumed pairwise disjoint), such that, for each* J:

(i) q_J *belongs to the subgroup of* Q_n *generated by the images of q under all deletions*;

(ii) q_J *is a product of left normed commutators of weight at least* $|J|$ *whose entries are from*

$$\{y_{ij}, y_{ij}^{-1} : j \in J,\ 1 \leqslant i \leqslant n\},$$

each containing at least one entry from $\{y_{ij}, y_{ij}^{-1}\}$ $(j \in J,\ 1 \leqslant i \leqslant n)$. (1.4.5)

The split-words q_J are all uniform and therefore

COROLLARY. *Each split-word is super-equivalent to a finite set of uniform split-words.* (1.4.6)

DEFINITION. *The* split-verbal sub-split-group *of a split-group* $\boldsymbol{G}$ *of species n*, determined *by* $S \subseteq \boldsymbol{Q}_n$, *is the sub-split-group* $S(\boldsymbol{G})$ *whose carrier is the subgroup of G generated by the set*

$$\{q\alpha : q \in S,\ \alpha : \boldsymbol{Q}_n \to \boldsymbol{G}\}. \tag{1.4.7}$$

Note that, by definition, this set admits every self-morphism of $\boldsymbol{G}$ hence so does the subgroup of G generated by it. In particular, this subgroup admits the splitting endomorphisms σ_i of $\boldsymbol{G}$ and hence carries a sub-split-group: so definition 1.4.7 is justified. Moreover, it follows that every split-verbal sub-split-group is fully invariant. As the carrier of $S(\boldsymbol{Q}_n)$ is the least subgroup to contain the images of S under all self-morphisms of $\boldsymbol{Q}_n$, the fully invariant closure of S in $\boldsymbol{Q}_n$ must contain $S(\boldsymbol{Q}_n)$; but as $S(\boldsymbol{Q}_n)$ is fully invariant and contains S, it follows that the fully invariant closure of S in $\boldsymbol{Q}_n$ is precisely $S(\boldsymbol{Q}_n)$:

THEOREM. *If* $S \subseteq \boldsymbol{Q}_n$, *then the fully invariant closure of S in* $\boldsymbol{Q}_n$ *is* $S(\boldsymbol{Q}_n)$. (1.4.8)

DEFINITION. *Two sets* S_1, S_2 *of split-words of the same species n are* equivalent *if they have the same normalized fully invariant closure in* $\boldsymbol{Q}_n$. (It is easily seen that the normal closure *qua* subgroup of a sub-split-group is a sub-split-group: If $\boldsymbol{U}$ is a sub-split-group of $\boldsymbol{G}$, $u \in U$, $g \in G$, then $(u^g)\,\sigma_i = (u\sigma_i)^{g\sigma_i} \in U^G$.) (1.4.9)

THEOREM. *If* S_1, S_2 *are super-equivalent, they are equivalent.* (1.4.10)

THEOREM. *Two sets* S_1, S_2 *of split-words of species n are equivalent if and only if the normalized split-verbal sub-split-groups they determine in every split-group of species n are equal.* (1.4.11)

Proof. One way around is obvious. For the other, suppose that S_1, S_2 are equivalent, and let $\boldsymbol{G}$ be any split-group of species n. We must show that

$$S_1(\boldsymbol{G})^G = S_2(\boldsymbol{G})^G.$$

The following lemma is useful here.

LEMMA. *If S is a set of split-words,* $\boldsymbol{G}$ *a split-group and* $\boldsymbol{N}$ a *normal sub-split-group of* $\boldsymbol{G}$, *then* $S(\boldsymbol{G}/\boldsymbol{N}) = S(\boldsymbol{G})\,\boldsymbol{N}/\boldsymbol{N}$. (1.4.12)

Proof. Every morphism $\alpha : \boldsymbol{Q}_n \to \boldsymbol{G}/\boldsymbol{N}$ can be factored through $\boldsymbol{G}$ via the natural morphism $\nu : \boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{N}$, say $\alpha = \beta\nu$. Conversely every $\beta : \boldsymbol{Q}_n \to \boldsymbol{G}$ can be continued to $\alpha : \boldsymbol{Q}_n \to \boldsymbol{G}/\boldsymbol{N}$ by $\alpha = \beta\nu$. Hence $S(\boldsymbol{G})\,\nu = S(\boldsymbol{G}/\boldsymbol{N})$ which is what we wanted.

The proof of (1.4.11) runs as follows. First note that if S is a subset of $\boldsymbol{Q}_n$ and $\alpha : \boldsymbol{Q}_n \to \boldsymbol{H}$, then $S(\boldsymbol{Q}_n)\,\alpha$ is contained in $S(\boldsymbol{H})$. Hence, with $\boldsymbol{H} = \boldsymbol{G}/\boldsymbol{N}$, we have $S_1(\boldsymbol{H}) = 1$ if and only if $S_1(\boldsymbol{Q}_n)$ is contained in $\cap\,\{\ker\alpha : \alpha : \boldsymbol{Q}_n \to \boldsymbol{H}\}$; and since S_1, S_2 are equivalent, this is true if and only if $S_2(\boldsymbol{Q}_n)$

is contained in $\cap\{\ker\alpha:\alpha:\boldsymbol{Q}_n\to\boldsymbol{H}\}$ which in turn is true if and only if $S_2(\boldsymbol{H}) = 1$. It follows that $S_1(\boldsymbol{G}) \leqslant S_2(\boldsymbol{G})^G$ (putting $\boldsymbol{N} = S_2(\boldsymbol{G})^G$) and therefore that $S_1(\boldsymbol{G})^G \leqslant S_2(\boldsymbol{G})^G$. In a similar way $S_2(\boldsymbol{G})^G \leqslant S_1(\boldsymbol{G})^G$, and this completes the proof.

Theorem 1.4.8 can be stated in a more familiar form for all relatively split-free split-groups as follows.

Theorem. *A sub-split-group of a relatively split-free split-group is fully invariant if and only if it is split-verbal.* (1.4.13)

Proof. Given a relatively split-free generating set of $\boldsymbol{G}$ and an element h of the sub-split-group $\boldsymbol{H}$ of $\boldsymbol{G}$, then there exists a finite subset T of that generating set such that $h\in\langle T\rangle$. There exists a finite subset T' of a free generating set of $\boldsymbol{Q}_n$ and a one-to-one map $\mu:T'\to T$ which extends to $\mu^*:\boldsymbol{Q}_n\to\boldsymbol{G}$.

Now $\langle T'\rangle\mu^* = \langle T\rangle$; hence there exists $q\in\langle T'\rangle$ with $q\mu^* = h$. Given $\alpha:\boldsymbol{Q}_n\to\boldsymbol{G}$ let $\beta:\boldsymbol{G}\to\boldsymbol{G}$ be an extension of $\mu^{-1}\alpha:T\to\boldsymbol{G}$. Then as $\mu^*\beta$ and α agree on T', they agree on $\langle T'\rangle$, hence, in particular, $q\alpha = q\mu^*\beta = h\beta\in\boldsymbol{H}$ if $\boldsymbol{H}$ is fully invariant. This proves that fully invariant sub-split-groups of $\boldsymbol{G}$ are split-verbal; and the converse is true in any split-group.

Theorem. *There is one-to-one correspondence between the (normalized) fully invariant sub-split-groups of $\boldsymbol{Q}_n/S(\boldsymbol{Q}_n)^{Q_n}$ and the (normalized) fully invariant sub-split-groups of $\boldsymbol{Q}_n$ containing $S(\boldsymbol{Q}_n)^{Q_n}$.* (1.4.14)

Proof. This proof is an easy application of the last theorem.

Lemma. *If $\boldsymbol{S}$ is a normal sub-split-group of $\boldsymbol{Q}_n$, then $S(\boldsymbol{G})$ is normal in $\boldsymbol{G}$ for all $\boldsymbol{G}$ of species n.* (1.4.15)

Proof. It is sufficient to show that $(q\alpha)^g\in S(\boldsymbol{G})$ whenever $q\in S$, $\alpha:\boldsymbol{Q}_n\to\boldsymbol{G}$, $g\in\boldsymbol{G}$. The proof is similar to that of (1.4.13); there exist $\alpha^*:\boldsymbol{Q}_n\to\boldsymbol{G}$, $\hat{g}\in\boldsymbol{Q}_n$ such that $q\alpha^* = q\alpha$, $\hat{g}\alpha^* = g$, so that

$$(q\alpha)^g = (q\alpha^*)^{\hat{g}\alpha^*} = (q^{\hat{g}})\,\alpha^*\in S(\boldsymbol{G})$$

since $\boldsymbol{S}$ is normal in $\boldsymbol{Q}_n$.

1.5. *Split-varieties*

Definition. *If S is a subset of $\boldsymbol{Q}_n$, the class of all split-groups $\boldsymbol{G}$ of species n such that $S(\boldsymbol{G}) = 1$ is the* variety of split-groups (*or, briefly, the* split-variety) *determined by S. For varieties of split-groups of small species we use special names; thus* bivariety *and* trivariety. (1.5.1)

Theorem. *Equivalent sets of split-words determine the same split-variety.* (1.5.2)

Proof. If S_1, S_2 are equivalent, then, by theorem 1.4.9, for any $\boldsymbol{G}$, $S_1(\boldsymbol{G})^G = 1$ if and only if $S_2(\boldsymbol{G})^G = 1$; that is $S_1(\boldsymbol{G}) = 1$ if and only if $S_2(\boldsymbol{G}) = 1$.

From this theorem it follows that, in defining split-varieties, we need only consider sets of split-words S which are normal, and carry fully invariant sub-split-groups of $\boldsymbol{Q}_n$, since every sub-set of $\boldsymbol{Q}_n$ is equivalent, by definition, to its normalized fully invariant closure. The normalized fully invariant closure of S is denoted by clS.

Definition. *If $\boldsymbol{S}$ is a normal, fully invariant sub-split-group of $\boldsymbol{Q}_n$, the split-variety determined by S will be denoted by $\mathscr{S}$. (Again we shall drop the prefix 'split' if the context allows.) We shall sometimes write $\mathscr{S}(\boldsymbol{G})$ and $\boldsymbol{S}(\boldsymbol{G})$ for $S(\boldsymbol{G})$.* (1.5.3)

Note that, for varieties of split-groups of species 1, that is to say, for varieties of groups in the usual sense, we will use the customary German letters.

Theorem. *The correspondence $\boldsymbol{S}\to\mathscr{S}$ between normal, fully invariant sub-split-groups $\boldsymbol{S}$ of $\boldsymbol{Q}_n$ and the varieties $\mathscr{S}$ of split-groups of species n is one-to-one and reverses inclusions.* (1.5.4)

Proof. Suppose $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ are normal and fully invariant in $\boldsymbol{Q}_n$ and $\mathscr{S}_1$ is contained in $\mathscr{S}_2$, then by lemma 1.4.10,

$$\boldsymbol{Q}_n/\boldsymbol{S}_1 \in \mathscr{S}_1 \leqslant \mathscr{S}_2,$$

and so $\boldsymbol{S}_2 = \boldsymbol{S}_2(\boldsymbol{Q}_n)$ is contained in $\boldsymbol{S}_1$. It follows that if $\mathscr{S}_1 = \mathscr{S}_2$, then $\boldsymbol{S}_1 = \boldsymbol{S}_2$.

It is clear that a split-variety is closed under the operations of forming sub-split-groups, quotient split-groups and cartesian products of split-groups. The converse of this is also true on account of theorem 1.2.7, and Birkhoff's corresponding result for varieties of universal algebras. The proof is omitted.

THEOREM. *A class of split-groups is closed under the operations of forming sub-split groups, quotient split-groups and cartesian products of split-groups if and only if it is a split-variety.* (1.5.5)

DEFINITION. *A split-word q in $\boldsymbol{Q}_n$ is a* split-law *in $\boldsymbol{G}$ if $\{q\}(\boldsymbol{G}) = 1$; simply written, $q(\boldsymbol{G}) = 1$. If $\mathscr{S}$ is a split-variety determined by the normal, fully invariant sub-split-group $\boldsymbol{S}$ of $\boldsymbol{Q}_n$ then the elements of $\boldsymbol{S}$ are called the split-laws of $\mathscr{S}$.* (1.5.6)

DEFINITION. *Given a split-variety $\mathscr{S}$ and an n-tuple $\boldsymbol{m} = (m_1, \ldots, m_n)$ such that $m_i = 0$ if $y_{i1} \in S$, we call $\boldsymbol{Q}(\boldsymbol{m})/S(\boldsymbol{Q}(\boldsymbol{m}))$ the split-free group $\boldsymbol{F}_m(\mathscr{S})$ of rank $\boldsymbol{m}$ of $\mathscr{S}$, and write $F_m(\mathscr{S})$ for its carrier.* (1.5.7)

By theorem 1.3.4 $\boldsymbol{F}_m(\mathscr{S})$ is relatively split-free of rank $\boldsymbol{m}$, and it lies in $\mathscr{S}$. Moreover,

THEOREM. *Every 'admissible' mapping of a split-free generating set of $\boldsymbol{F}_m(\mathscr{S})$ into a split-group $\boldsymbol{G}$ in $\mathscr{S}$ can be extended to a morphism.* (1.5.8)

Proof. Let $\boldsymbol{z} = \{z_{ij}: j \in J_i,\ 1 \leqslant i \leqslant n\}$ be a split-free generating set for $\boldsymbol{Q}(\boldsymbol{m})$; then if $\nu: \boldsymbol{Q}(\boldsymbol{m}) \to \boldsymbol{F}_m(\mathscr{S})$ is the natural morphism, $\{z_{ij}\nu: j \in J_i,\ 1 \leqslant i \leqslant n\} = \boldsymbol{z}\nu$ is a split-free generating set of $\boldsymbol{F}_m(\mathscr{S})$. Suppose $\beta: \boldsymbol{z}\nu \to \boldsymbol{G} \in \mathscr{S}$ such that $z_{ij}\nu\beta$ is in $A_i(\boldsymbol{G})$. Then $\nu\beta: \boldsymbol{z} \to \boldsymbol{G}$ extends to a morphism $\delta: \boldsymbol{Q}(\boldsymbol{m}) \to \boldsymbol{G}$. Since $\boldsymbol{Q}(\boldsymbol{m})\,\delta$ is contained in $\boldsymbol{G}$ and $\boldsymbol{G}$ is in $\mathscr{S}$ it follows easily that ker δ contains $S(\boldsymbol{Q}(\boldsymbol{m}))$. Hence δ can be factored through ν, say $\delta = \nu\gamma$ and by definition, $(\boldsymbol{z}\nu)\,\delta = (\boldsymbol{z}\nu)\,\beta$; γ is the extension of β. This completes the proof.

Theorem 1.3.4 yields,

THEOREM. *Every relatively split-free split-group is split-free in some $\mathscr{S}$.* (1.5.9)

THEOREM. *If $\boldsymbol{S}$ is a fully invariant sub-split-group of $\boldsymbol{Q}_n$ then $\boldsymbol{S}(\boldsymbol{m}) = \boldsymbol{S} \cap \boldsymbol{Q}(\boldsymbol{m})$ is fully invariant in $\boldsymbol{Q}(\boldsymbol{m})$, and*

$$\boldsymbol{S}(\boldsymbol{m}) = S(\boldsymbol{Q}(\boldsymbol{m})),$$

and
$$\boldsymbol{S}(\boldsymbol{m})^{Q(m)} = \boldsymbol{S}^{Q_n} \cap \boldsymbol{Q}(\boldsymbol{m}). \tag{1.5.10}$$

Proof. Clearly $S(\boldsymbol{Q}(\boldsymbol{m}))$ is contained in $\boldsymbol{Q}(\boldsymbol{m}) \cap \boldsymbol{S} = \boldsymbol{S}(\boldsymbol{m})$. Conversely, if q is in $\boldsymbol{Q}(\boldsymbol{m}) \cap \boldsymbol{S}$ and α is a self-morphism of $\boldsymbol{Q}_n$ which maps $\boldsymbol{Q}(\boldsymbol{m})$ identically and everything else to 1, then $q = q\alpha \in S(\boldsymbol{Q}(\boldsymbol{m}))$ which gives us the opposite inequality.

For the second part we have

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{m})^{Q(m)} &= (S \cap \boldsymbol{Q}(\boldsymbol{m}))^{Q(m)} \leqslant \boldsymbol{S}^{Q_n} \cap \boldsymbol{Q}(\boldsymbol{m}),\\ &= S^{Q_n}(\boldsymbol{Q}(\boldsymbol{m})), \quad \text{by the first part},\\ &\leqslant S(\boldsymbol{Q}(\boldsymbol{m}))^{Q(m)},\\ &= \boldsymbol{S}(\boldsymbol{m})^{Q(m)},\end{aligned}$$

again by the first part.

DEFINITION. *The split-variety* generated *by a set* $\{G_i : i \in I\}$ *of split-groups of the same species n is the smallest split-variety of species n which contains all* G_i; *equivalently, the split-variety generated by* $\{G_i : i \in I\}$ *is the class of split-groups satisfying the split-laws which hold in all* G_i. *We denote this split-variety by* svar $(\{G_i : i \in I\})$. (1.5.11)

DEFINITION. *The* join *of two split-varieties* $\mathscr{S}, \mathscr{T}$ *of the same species is the split-variety generated by the set* $\{G_i : G_i \in \mathscr{S}$ *or* $G_i = \mathscr{T}\}$; *the* meet *of* $\mathscr{S}, \mathscr{T}$ *is the class intersection of* $\mathscr{S}, \mathscr{T}$. *We denote join and meet by* $\mathscr{S} \vee \mathscr{T}$ *and* $\mathscr{S} \wedge \mathscr{T}$ *respectively. The commutator split-variety* $[\mathscr{S}, \mathscr{T}]$ *is that one determined by* $[S, T]$. (1.5.12)

THEOREM. *The laws of* $\mathscr{S} \vee \mathscr{T}$, $\mathscr{S} \wedge \mathscr{T}$ *are* $S \cap T$ *and* ST *respectively.* (1.5.13)

Proof. The proofs follow easily from the definitions and we omit them.

THEOREM. *A split-variety* $\mathscr{S}$ *is generated by its finitely generated split-groups.* (1.5.14)

Proof. If $\mathscr{T}$ is the sub-split-variety generated by the finitely generated split-groups of $\mathscr{S}$, let q be a split-law of $\mathscr{T}$, H belong to $\mathscr{S}$, and $\alpha: Q_n \to H$. As previously, we may suppose that α acts non-trivially on only finitely many free generators y_{ij}, so that $Q_n\alpha (\leqslant H)$ is finitely generated, and therefore $q\alpha = 1$; whence $\mathscr{S} = \mathscr{T}$.

1.6. *Products of split-varieties*

In this section we introduce a product operation on the variety of all split-groups of species n, imitating exactly the familiar definition for species 1 (Hanna Neumann 1967, 21.11).

DEFINITION. *If* $\mathscr{S}, \mathscr{T}$ *are split-varieties of species n then their product* $\mathscr{S}\mathscr{T}$ *is the split-variety consisting of all split-groups* G *of species n which contain a normal sub-split-group* N *in* $\mathscr{S}$ *such that* G/N *is in* $\mathscr{T}$. (1.6.1)

It is easy to check that $\mathscr{S}\mathscr{T}$ is indeed a split-variety (use (1.5.5)).

LEMMA. *If* $\mathscr{S}, \mathscr{T}$ *are split-varieties of species n and* G *has species n, then* $S(T(G)) = S(T)(G)$. (1.6.2)

Proof. Let q belong to S, and $\alpha: Q_n \to T(G)$. Let y denote a finite set of y_{ij}'s such that $q \in \langle y \rangle$. There exists $\beta: Q_n \to G$, and to each y_{ij} in y some q_{ij} in T such that $y_{ij}\alpha = q_{ij}\beta$. Hence, defining $\gamma: Q_n \to Q_n$ by $y_{ij}\gamma = q_{ij}$ for y_{ij} from y, and arbitrarily elsewhere, we have

$$q\alpha = q\gamma\beta = (q\gamma)\beta.$$

It follows that $S(T(G))$ is contained in $S(T)(G)$. The opposite inclusion is proved in a similar manner.

The next two theorems are immediate consequences of this lemma.

THEOREM. *Multiplication of split-varieties is associative.* (1.6.3)

THEOREM. *The split-laws of* $\mathscr{S}\mathscr{T}$ *are precisely* $S(T)$. (1.6.4)

One naturally asks if the product, as here defined, has the same distributive behaviour for all species as for species 1 (21.22 to 21.25 in Hanna Neumann 1967). In order to discuss this we need lemma 1.6.6 below, for which (1.6.5) is preparatory.

LEMMA. *Let* N *be a non-trivial normal sub-bigroup of* Q_2. *Then there exists a subgroup* C *of* $A_1(N)$ *of rank* ∞ *such that*

$$N = C * A_2(N). \qquad (1.6.5)$$

Proof. First note that, without loss of generality, we may assume that $A_2(\mathbf{Q}_2)$ is contained in N. For, if T is a complete set of left coset representatives for $N \cap A_2(\mathbf{Q}_2)$ in $A_2(\mathbf{Q}_2)$, we have

$$A_1(\mathbf{Q}_2) = \Pi^*\{Y_1^b : b \in A_2(\mathbf{Q}_2)\}$$
$$= \Pi^*\{\Pi^*\{Y_1^t : t \in T\}^b : b \in N \cap A_2(\mathbf{Q}_2)\}$$

and so, putting $\hat{Y}_1 = \Pi^*\{Y_1^t : t \in T\}$, we have

$$N \leqslant A_1(\mathbf{Q}_2) \,.\, N \cap A_2(\mathbf{Q}_2) = \hat{\mathbf{Y}}_1 * (N \cap A_2(\mathbf{Q}_2)),$$

which carries an isomorphic copy of $\mathbf{Q}_2$.

Under this assumption it follows that N contains $[Y_1, Y_2]$ and hence that a complete set of right coset representatives for $A_1(\mathbf{N})$ in Q_2 is

$$U = \{bs : b \in Y_2,\ s \in S\},$$

where S is a complete set of right coset representatives for $N \cap Y_1$ in Y_1; further if S is a Schreier system (see Marshall Hall (1956, ch. 7)), so is U, and in this case we can write down a free generating set for $A_1(\mathbf{N})$ as follows. Let $\phi(q)$ denote the element of U which represents q, and then $A_1(\mathbf{N})$ is freely generated by the non-trivial elements in the set

$$\{ux\phi(ux)^{-1} : u \in U,\ x \in \{y_{1i}, y_{2i} : i \in I^+\}\}.$$

Now, for b in Y_2 and s in S we have $\phi(bsy_{1i}) = b\psi(sy_{1i})$ (where $\psi(sy_{1i})$ is the element of S representing sy_{1i}); and $\phi(bsy_{2i}) = (by_{2i})\,s$. It follows that $A_1(\mathbf{N})$ is freely generated by the non-trivial elements of

$$\{b(sy_{1i}\psi(sy_{1i})^{-1})\,b^{-1},\ b(sy_{2i}s^{-1}y_{2i}^{-1})\,b^{-1} : b \in Y_2,\ s \in S,\ i \in I^+\}.$$

If we put $C = \langle sy_{1i}\psi(sy_{1i})^{-1}, [s^{-1}, y_{2i}^{-1}] : s \in S,\ i \in I^+\rangle$ then we have

$$A_1(\mathbf{N}) = \Pi^*\{C^b : b \in Y_2\},$$

whence $N = A_1(\mathbf{N})\,A_2(\mathbf{N}) = C * Y_2$ as required. Clearly C has rank ∞.

Lemma. *Suppose that* $\mathbf{N}$ *is a non-trivial normal sub-bigroup of* $\mathbf{Q}_n$. *Then, if* m *is the least natural number such that* $N \cap B_{m+1}(\mathbf{Q}_n) = 1$, $\mathbf{N} \cong \mathbf{Q}_m$. (1.6.6)

Proof. Notice first that if we forget the split-group structure on $B_2(\mathbf{Q}_n)$, then Q_n can be thought of as carrying a bigroup $\mathbf{B}$, isomorphic to $\mathbf{Q}_2$, in which $A_1(\mathbf{B}) = A_1(\mathbf{Q}_n)$ and $A_2(\mathbf{B}) = B_2(\mathbf{Q}_n)$. The previous lemma then yields the existence of C_1 contained in $A_1(\mathbf{N})$ such that

$$N = C_1 * (N \cap B_2(\mathbf{Q}_n)),$$

and with rank ∞. We may now assume inductively that there exist C_i contained in

$$(N \cap B_2(\mathbf{Q}_n)) \cap A_i(\mathbf{Q}_n) = A_i(\mathbf{N})\ (2 \leqslant i \leqslant m)$$

with rank ∞ such that

$$N \cap B_2(\mathbf{Q}_n) = C_2 * \ldots * C_m.$$

This completes the proof.

Theorem. $\mathscr{S}_1 \leqslant \mathscr{S}_2$ *implies* $\mathscr{S}_1\mathscr{T} \leqslant \mathscr{S}_2\mathscr{T}$. *Conversely, if* $T \cap A_n(\mathbf{Q}_n)$ *is non-trivial, then* $\mathscr{S}_1\mathscr{T} \leqslant \mathscr{S}_2\mathscr{T}$ *implies* $\mathscr{S}_1 \leqslant \mathscr{S}_2$; *in particular if* $\mathscr{S}_1\mathscr{T} = \mathscr{S}_2\mathscr{T}$ *then* $\mathscr{S}_1 = \mathscr{S}_2$. (1.6.7)

Proof. Under the conditions imposed on $\mathscr{T}$, $\mathbf{T} \cong \mathbf{Q}_n$ by lemma 1.6.6.

Theorem.
(i) $(\mathscr{S}_1 \vee \mathscr{S}_2)\,\mathscr{T} = \mathscr{S}_1\mathscr{T} \vee \mathscr{S}_2\mathscr{T}$,
(ii) $(\mathscr{S}_1 \wedge \mathscr{S}_2)\,\mathscr{T} = \mathscr{S}_1\mathscr{T} \wedge \mathscr{S}_2\mathscr{T}$,
(iii) $\mathscr{S}(\mathscr{T}_1 \vee \mathscr{T}_2) \geqslant \mathscr{S}\mathscr{T}_1 \vee \mathscr{S}\mathscr{T}_2$,
and (iv) $\mathscr{S}(\mathscr{T}_1 \wedge \mathscr{T}_2) \leqslant \mathscr{S}\mathscr{T}_1 \wedge \mathscr{S}\mathscr{T}_2$,

and the last two inclusions may be proper. (1.6.8)

Proof. (i) Suppose that m is the least natural number such that $T \cap B_{m+1}(\boldsymbol{Q}_n) = 1$. Then by (1.6.6), $\boldsymbol{T} \cong \boldsymbol{Q}_m$ and

$$\begin{aligned}(S_1 \cap S_2)(\boldsymbol{T}) &= (S_1 \cap S_2 \cap Q_m)(\boldsymbol{T}) \\ &= (S_1 \cap Q_m)(\boldsymbol{T}) \cap (S_2 \cap Q_m)(\boldsymbol{T}) = S_1(\boldsymbol{T}) \cap S_2(\boldsymbol{T}).\end{aligned}$$

Theorem 1.6.4 then gives the result we want.

(ii) Clearly $(\mathscr{S}_1 \wedge \mathscr{S}_2)\mathscr{T}$ is contained in $\mathscr{S}_1\mathscr{T} \wedge \mathscr{S}_2\mathscr{T}$. For the converse direction let $\boldsymbol{G}$ belong to $\mathscr{S}_1\mathscr{T} \wedge \mathscr{S}_2\mathscr{T}$ so that there exist $\boldsymbol{N}_i$, normal in $\boldsymbol{G}$, with $\boldsymbol{N}_i$ in $\mathscr{S}_i$ and $\boldsymbol{G}/\boldsymbol{N}_i$ in $\mathscr{T}$ $(i = 1, 2)$. Hence $\boldsymbol{N}_1 \cap \boldsymbol{N}_2$ is in $\mathscr{S}_1 \wedge \mathscr{S}_2$ and therefore $\boldsymbol{G}/\boldsymbol{N}_1 \cap \boldsymbol{N}_2$ is a subdirect product of $\boldsymbol{G}/\boldsymbol{N}_1$ and $\boldsymbol{G}/\boldsymbol{N}_2$, both of which belong to $\mathscr{T}$; that is, $\boldsymbol{G}$ is in $(\mathscr{S}_1 \wedge \mathscr{S}_2)\mathscr{T}$.

(iii) to (iv) The inclusions shown are immediate; equality fails even for species 1 (examples 8.1, 8.2 in Neumann, Neumann & Neumann 1962).

In terms of the product defined within species n, we can define a product of split-varieties of arbitrary species, the 'circle' product, and this is done in (1.6.9) below. Its advantage lies in the fact that it provides a framework for relating split-varieties to products of varieties of groups, as well as providing a convenient formalism.

As a temporary piece of notation write $\mathscr{S}^{(n)}$ for the variety of split-groups $\boldsymbol{G}$ of species $m+n$ (where $\mathscr{S}$ has species m) such that $A_{m+1}(\boldsymbol{G}) = \ldots = A_{m+n}(\boldsymbol{G}) = 1$ and $(G, A_1(\boldsymbol{G}), \ldots, A_m(\boldsymbol{G}))$ belongs to $\mathscr{S}$. Similarly, ${}^{(n)}\mathscr{S}$ denotes the variety of split-groups $\boldsymbol{G}$ of species $m+n$ such that $A_1(\boldsymbol{G}) = \ldots = A_n(\boldsymbol{G}) = 1$ and $(G, A_{n+1}(\boldsymbol{G}), \ldots, A_{m+n}(\boldsymbol{G}))$ belongs to $\mathscr{S}$.

Definition. *If $\mathscr{S}$, $\mathscr{T}$ have species m, n respectively we write* $\mathscr{S} \circ \mathscr{T} = \mathscr{S}^{(n)}\,{}^{(m)}\mathscr{T}$. (1.6.9)

Lemma. *If $\mathscr{S}_1$, $\mathscr{S}_2$ have species m, then for each natural number n*

$$(\mathscr{S}_1 \vee \mathscr{S}_2)^{(n)} = \mathscr{S}_1^{(n)} \vee \mathscr{S}_2^{(n)}, \quad (\mathscr{S}_1 \wedge \mathscr{S}_2)^{(n)} = \mathscr{S}_1^{(n)} \wedge \mathscr{S}_2^{(n)},$$
$${}^{(n)}(\mathscr{S}_1 \vee \mathscr{S}_2) = {}^{(n)}\mathscr{S}_1 \vee {}^{(n)}\mathscr{S}_2 \quad \textit{and} \quad {}^{(n)}(\mathscr{S}_1 \wedge \mathscr{S}_2) = {}^{(n)}\mathscr{S}_1 \wedge {}^{(n)}\mathscr{S}_2. \qquad (1.6.10)$$

The proofs of these facts are straightforward and we omit them.

Theorem. *$\mathscr{S} \circ \mathscr{T}$ consists of all those split-groups $\boldsymbol{G}$ such that (the split-group of species m carried by) $A_1(\boldsymbol{G}) \ldots A_m(\boldsymbol{G})$ is in $\mathscr{S}$ and (the split-group of species n carried by) $A_{m+1}(\boldsymbol{G}) \ldots A_{m+n}(\boldsymbol{G})$ is in $\mathscr{T}$.* (1.6.11)

Proof. $\boldsymbol{G}$ is in $\mathscr{S} \circ \mathscr{T}$ if and only if $\boldsymbol{G}$ has a normal sub-bigroup $\boldsymbol{N}$ in $\mathscr{S}^{(n)}$ with $\boldsymbol{G}/\boldsymbol{N}$ in ${}^{(m)}\mathscr{T}$. Now $\boldsymbol{N}$ is in $\mathscr{S}^{(n)}$ if and only if $A_i(\boldsymbol{N}) = 1$ for $m+1 \leqslant i \leqslant n$; and $\boldsymbol{G}/\boldsymbol{N}$ is in ${}^{(m)}\mathscr{T}$ if and only if $A_1(\boldsymbol{G}) \ldots A_m(\boldsymbol{G})$ is contained in $\boldsymbol{N}$. Hence $\boldsymbol{G}$ is in $\mathscr{S} \circ \mathscr{T}$ if and only if $N = A_1(\boldsymbol{G}) \ldots A_m(\boldsymbol{G})$ and the result follows from this.

The split-laws of $\mathscr{S} \circ \mathscr{T}$ may be described as follows. Let ξ be the endomorphism of Q_{m+n} defined by

$$\begin{aligned} y_{ij}\xi &= y_{i+mj}, \quad 1 \leqslant i \leqslant n, j \in I^+, \\ y_{ij}\xi &= 1, \qquad n+1 \leqslant i, j \in I^+. \end{aligned}$$

Then, recalling that $\boldsymbol{Q}_m$, $\boldsymbol{Q}_n$ are naturally embedded in $\boldsymbol{Q}_{m+n}$ we have

Theorem. *The split-laws of $\mathscr{S} \circ \mathscr{T}$ are $cl\{S \cup T\xi\} = S(\boldsymbol{Q}_{m+n})\, T\xi(\boldsymbol{Q}_{m+n})$.* (1.6.12)

Proof. The split-laws of $\mathscr{S} \circ \mathscr{T}$ are $S^{(n)}({}^{(m)}\boldsymbol{T})$, ${}^{(m)}T = A_1(\boldsymbol{Q}_{m+n}) \ldots A_m(\boldsymbol{Q}_{m+n}) \,.\, T\xi$, and therefore

$$\begin{aligned} S^{(n)}({}^{(m)}\boldsymbol{T}) &= S(A_1(\boldsymbol{Q}_{m+n}) \ldots A_m(\boldsymbol{Q}_{m+n}))\, T\xi(\boldsymbol{Q}_{m+n}) \\ &= S(\boldsymbol{Q}_{m+n})\, T\xi(\boldsymbol{Q}_{m+n}) \\ &= cl(S \cup T\xi). \end{aligned}$$

Theorem. *The 'circle' product is associative.* (1.6.13)

Proof. Use (1.6.11).

Finally in this section we note the distributive behaviour of the circle product.

Theorem. (i) $(\mathscr{S}_1 \vee \mathscr{S}_2) \circ \mathscr{T} = \mathscr{S}_1 \circ \mathscr{T} \vee \mathscr{S}_2 \circ \mathscr{T}$;

(ii) $(\mathscr{S}_1 \wedge \mathscr{S}_2) \circ \mathscr{T} = \mathscr{S}_1 \circ \mathscr{T} \wedge \mathscr{S}_2 \circ \mathscr{T}$;

(iii) $\mathscr{S} \circ (\mathscr{T}_1 \wedge \mathscr{T}_2) = \mathscr{S} \circ \mathscr{T}_1 \wedge \mathscr{S} \circ \mathscr{T}_2$.

By contrast (iv) $\mathscr{S} \circ (\mathscr{T}_1 \vee \mathscr{T}_2) \geqslant \mathscr{S} \circ \mathscr{T}_1 \vee \mathscr{S} \circ \mathscr{T}_2$,

with equality if and only if $\mathscr{T}_1$, $\mathscr{T}_2$ *are comparable (in the case* $\mathscr{S} \neq \mathfrak{E} \circ \ldots \circ \mathfrak{E}$). (1.6.14)

Proof. (i) follows from (1.6.10) and (1.6.8); (ii) to (iv) are immediate, the only non-trivial thing being to check when equality holds in (iv). Clearly $\mathscr{T}_1$, $\mathscr{T}_2$ comparable is sufficient for equality; to prove necessity let q_1 be in T_1, q_2 be in T_2 and then $\mathscr{S} \circ \mathscr{T}_1 \vee \mathscr{S} \circ \mathscr{T}_2$ has a split-law

$$[y_1, q_1 \xi, q_2 \xi],$$

which is therefore a split-law in the split-group $\boldsymbol{W}$ of species $m+n$ carried by the group $W = F_\infty(\mathscr{S}) \operatorname{wr} (F_\infty(\mathscr{T}_1) \times F_\infty(\mathscr{T}_2))$ in the natural way. If $\mathscr{T}_1$ is not contained in $\mathscr{T}_2$ then, by virtue of 24.22 in Hanna Neumann (1967), $[y_1, q_1 \xi]$ is a split-law in $\boldsymbol{W}$, and therefore, for the same reason, $\mathscr{S}$ non-trivial implies $q_1 \xi$ is a split-law in $\boldsymbol{W}$; hence q_1 is a split-law in $\boldsymbol{F}_\infty(\mathscr{T}_2)$. Since q_1 is an arbitrary element of T_1 this means $\mathscr{T}_2$ is contained in $\mathscr{T}_1$.

1.7. *Split-varieties and products of varieties of groups*

For each split-variety $\mathscr{S}$ define the variety $\mathscr{S}\tau$ by

$$\mathscr{S}\tau = \operatorname{var}\{G : \boldsymbol{G} \in \mathscr{S}\}.$$

If $\boldsymbol{S}$ has species n and $\mathfrak{B}_i = \operatorname{var}\{A_i(\boldsymbol{G}) : \boldsymbol{G} \in \mathscr{S}\}$ $(1 \leqslant i \leqslant n)$, then $\mathscr{S}\tau$ is a subvariety of $\mathfrak{B}_1 \mathfrak{B}_2 \ldots \mathfrak{B}_n$. Conversely, if $\mathfrak{W} = \mathfrak{B}_1 \mathfrak{B}_2 \ldots \mathfrak{B}_n$ is a given product variety and $\mathfrak{U}$ is a subvariety of $\mathfrak{W}$, define

$$\mathfrak{U}\sigma = \{(G, A_1, \ldots, A_n) : G \in \mathfrak{U},\ A_i \in \mathfrak{B}_i,\ 1 \leqslant i \leqslant n\}.$$

Note that, by virtue of (1.6.11),

$$\mathfrak{W}\sigma = \mathfrak{B}_1 \circ \mathfrak{B}_2 \circ \ldots \circ \mathfrak{B}_n. \tag{1.7.1}$$

From now on we will assume that $\mathfrak{W} = \mathfrak{B}_1 \mathfrak{B}_2 \ldots \mathfrak{B}_n$ is given and fixed, and regard τ as a mapping $\tau : \Lambda(\mathfrak{B}_1 \circ \ldots \circ \mathfrak{B}_n) \to \Lambda(\mathfrak{W})$ (see definition 4.1.2). Notice that σ depends very much on $\mathfrak{W}$ and on the decomposition exhibited: for example, if $\mathfrak{W} = \mathfrak{A}_2 \mathfrak{A}_2 \mathfrak{A}_2$ and $\boldsymbol{G}$ is the bigroup whose carrier G is dihedral of order 8, $A_1(\boldsymbol{G})$ the normal 4-cycle and $A_2(\boldsymbol{G})$ some 2-cycle, then $\boldsymbol{G}$ belongs to $(\mathfrak{A}_2 \mathfrak{A}_2) \circ \mathfrak{A}_2$ but not to $\mathfrak{A}_2 \circ (\mathfrak{A}_2 \mathfrak{A}_2)$.

Lemma. (i) *If* $\mathfrak{U} \leqslant \mathfrak{W}$, $\mathscr{S} \leqslant \mathfrak{W}\sigma$ *then* $\mathfrak{U}\sigma\tau \leqslant \mathfrak{U}$ *and* $\mathscr{S}\tau\sigma \geqslant \mathscr{S}$.

(ii) $\sigma\tau\sigma = \sigma$, $\tau\sigma\tau = \tau$. (1.7.2)

Proof. (i) is immediate from the definitions of σ and τ, and for (ii) note that for all subvarieties $\mathfrak{U}$ of $\mathfrak{W}$ (i) yields

$$\mathfrak{U}\sigma \leqslant (\mathfrak{U}\sigma)\tau\sigma = (\mathfrak{U}\sigma\tau)\sigma \leqslant \mathfrak{U}\sigma,$$

from which $\sigma\tau\sigma = \sigma$ follows. The other part follows similarly.

Corollary. $(\sigma\tau)^2 = \sigma\tau, \quad (\tau\sigma)^2 = \tau\sigma.$ (1.7.3)

Definition. (i) *A subvariety* $\mathfrak{U}$ *of* $\mathfrak{W}$ *is* closed *if and only if* $\mathfrak{U}\sigma\tau = \mathfrak{U}$; *a sub-split-variety* $\mathscr{S}$ *of* $\mathfrak{W}\sigma$ *is* open *if and only if* $\mathscr{S}\tau\sigma = \mathscr{S}$.

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(ii) *The* interior *of a subvariety* $\mathfrak{U}$ *of* $\mathfrak{W}$ *is* $\mathfrak{U}\sigma\tau$; *the* exterior *of a sub-split-variety* $\mathscr{S}$ *of* $\mathfrak{W}\sigma$ *is* $\mathscr{S}\tau\sigma$. (1.7.4)

Because σ depends on the decomposition of $\mathfrak{W}$ chosen it is clear that all the terms defined in (1.7.4) are relative. From (1.7.3) it follows that interiors are closed and exteriors are open. Indeed, (1.7.2) (ii) yields

COROLLARY. *Every* $\mathfrak{U}\sigma$ *is open and every* $\mathscr{S}\tau$ *is closed.* (1.7.5)

THEOREM. *The mapping* σ *is a meet-homomorphism and the mapping* τ *is a join-homomorphism*; *and* $\tau\sigma$ *need be neither a meet-, nor a join-, homomorphism. In particular* σ *may not be a join-homomorphism and* τ *may not be a meet-homomorphism.* (1.7.6)

Proof. It is evident at once from the definitions that σ is a meet-, and τ a join-, homomorphism. (This will be brought out later also when we describe the laws and split-laws of $\mathscr{S}\tau$ and $\mathfrak{U}\sigma$ in (1.7.15).) The following examples demonstrate the other statements; in each case let $\sigma_0 : \Lambda(\mathfrak{A}\mathfrak{A}) \to \Lambda(\mathfrak{A} \circ \mathfrak{A})$.

EXAMPLE. $\tau\sigma_0$ *is not a meet-homomorphism.* (1.7.7)

Note first that $\mathfrak{A}_2\mathfrak{A}_2\sigma_0 < (\mathfrak{A}_2\mathfrak{A} \wedge \mathfrak{A}\mathfrak{A}_2)\sigma_0$; for if

$$G = \langle a, b : a^8 = b^2 = 1, a^b = a^5\rangle \quad \text{then} \quad (G, \langle a\rangle, \langle b\rangle) \in (\mathfrak{A}_2\mathfrak{A} \wedge \mathfrak{A}\mathfrak{A}_2)\sigma_0 - \mathfrak{A}_2\mathfrak{A}_2\sigma_0.$$

Hence

$$\begin{aligned}(\mathfrak{A}_2 \circ \mathfrak{A} \wedge \mathfrak{A} \circ \mathfrak{A}_2)\tau\sigma_0 = \mathfrak{A}_2 \circ \mathfrak{A}_2\tau\sigma_0 = \mathfrak{A}_2\mathfrak{A}_2\sigma_0 &< (\mathfrak{A}_2\mathfrak{A} \wedge \mathfrak{A}\mathfrak{A}_2)\sigma_0 \\ &= \mathfrak{A}_2\mathfrak{A}\sigma_0 \wedge \mathfrak{A}\mathfrak{A}_2\sigma_0 = \mathfrak{A}_2 \circ \mathfrak{A}\tau\sigma_0 \wedge \mathfrak{A} \circ \mathfrak{A}_2\tau\sigma_0.\end{aligned}$$

EXAMPLE. $\tau\sigma_0$ *is not a join-homomorphism.* (1.7.8)

First, $(\mathfrak{A}_2\mathfrak{A} \vee \mathfrak{A}\mathfrak{A}_2)\sigma_0 > \mathfrak{A}_2\mathfrak{A}\sigma_0 \vee \mathfrak{A}\mathfrak{A}_2\sigma_0$: we show that the bigroup $\boldsymbol{H}$ carried by

$$C_4 \,\mathrm{wr}\, C_4/(C_4 \,\mathrm{wr}\, C_4)_{(6)}$$

in the natural way belongs to $(\mathfrak{A}_2\mathfrak{A} \vee \mathfrak{A}\mathfrak{A}_2)\sigma_0$ but not to $\mathfrak{A}_2\mathfrak{A}\sigma_0 \vee \mathfrak{A}\mathfrak{A}_2\sigma_0$. For, $\mathfrak{A}_2\mathfrak{A}\sigma_0 \vee \mathfrak{A}\mathfrak{A}_2\sigma_0$ has a bilaw $[y_1^2, z_1^2]$ $(= [y_1, z_1^2]^2)$, and it is easy to verify that $\boldsymbol{H}$ does not; and a special case of an unpublished result by L. G. Kovács, namely

$$\mathfrak{A}^2 \wedge \mathfrak{A}_2\mathfrak{A}\mathfrak{A}_2 = \mathfrak{A}_2\mathfrak{A} \vee \mathfrak{A}\mathfrak{A}_2,$$

makes it easy to show that H is in $\mathfrak{A}_2\mathfrak{A} \vee \mathfrak{A}\mathfrak{A}_2$.

Finally,

$$\begin{aligned}(\mathfrak{A}_2 \circ \mathfrak{A} \vee \mathfrak{A} \circ \mathfrak{A}_2)\tau\sigma_0 &= (\mathfrak{A}_2 \circ \mathfrak{A}\tau \vee \mathfrak{A} \circ \mathfrak{A}_2\tau)\sigma_0 \\ &= (\mathfrak{A}_2\mathfrak{A} \vee \mathfrak{A}\mathfrak{A}_2)\sigma_0 > \mathfrak{A}_2\mathfrak{A}\sigma_0 \vee \mathfrak{A}\mathfrak{A}_2\sigma_0 = \mathfrak{A}_2 \circ \mathfrak{A}\tau\sigma_0 \vee \mathfrak{A} \circ \mathfrak{A}_2\tau\sigma_0.\end{aligned}$$

It is unlikely that $\sigma\tau$ is a homomorphism in general, but I have been unable to produce examples to show this.

THEOREM. *The following conditions are equivalent*:

(i) σ *is one-to-one*;

(ii) *every subvariety* $\mathfrak{U}$ *of* $\mathfrak{W}$ *is closed*;

(iii) τ *is onto.* (1.7.9)

Proof. From (1.7.2) (ii), $(\mathfrak{U}\sigma\tau)\sigma = \mathfrak{U}\sigma$ so that if σ is one-to-one then $\mathfrak{U}\sigma\tau = \mathfrak{U}$; that is, (i) implies (ii). Clearly (ii) implies (iii). Finally, if τ is onto, and $\mathfrak{U}_1\sigma = \mathfrak{U}_2\sigma$, write $\mathscr{S}_1\tau = \mathfrak{U}_1$, $\mathscr{S}_2\tau = \mathfrak{U}_2$ and then $\mathscr{S}_1\tau\sigma\tau = \mathscr{S}_2\tau\sigma\tau$ so that $\mathscr{S}_1\tau = \mathscr{S}_2\tau$, or $\mathfrak{U}_1 = \mathfrak{U}_2$.

In an entirely similar manner one proves also

THEOREM. *The following conditions are equivalent*:

(i) σ *is onto*;

(ii) *every subvariety* $\mathscr{S}$ *of* $\mathfrak{W}\sigma$ *is open*;

(iii) τ *is one-to-one.* (1.7.10)

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EXAMPLE. *If* $\sigma_0 : \Lambda(\mathfrak{A}\mathfrak{A}) \to \Lambda(\mathfrak{A} \circ \mathfrak{A})$ *then neither* σ_0 *nor* τ *is one-to-one or onto.* (1.7.11)

For $\mathfrak{E} \circ \mathfrak{A}\tau = \mathfrak{A} \circ \mathfrak{E}\tau$ so that τ is not one-to-one and σ_0 is not onto. To show that σ_0 is not one-to-one is more difficult, and we use the description of $\Lambda(\mathfrak{A}_p\mathfrak{A}_p)$, p prime, provided by Newman & Kovács (1970); they show, *inter alia*, that the subvariety $\mathfrak{U}_\lambda$ of $\mathfrak{A}_p\mathfrak{A}_p$ determined by the additional law $\prod_{i=2}^{\lambda p}[x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\lambda p}]$ $(\lambda \geqslant 2)$ properly contains $\mathfrak{A}_p\mathfrak{A}_p \wedge \mathfrak{N}_{\lambda p-1}$. It is easy to verify that if G is in $\mathfrak{U}_\lambda\sigma_0$ then G is in $\mathfrak{A}_p\mathfrak{A}_p \wedge \mathfrak{N}_{\lambda p-1}$ and so $\mathfrak{U}_\lambda\sigma_0 = (\mathfrak{A}_p\mathfrak{A}_p \wedge \mathfrak{N}_{\lambda p-1})\sigma_0$. In other words σ_0 is not one-to-one and, by (1.7.9) τ is not onto.

The remainder of this section will be concerned with relationships between the free groups of subvarieties $\mathfrak{U}$ of $\mathfrak{W}$ and the free split-groups of $\mathfrak{U}\sigma$, and also with that between free split-groups of $\mathscr{S}(\leqslant \mathfrak{W}\sigma)$ and free groups of $\mathscr{S}\tau$.

THEOREM. *The split-free split-group of rank* $(m_1, \ldots, m_n)$ *of* $\mathfrak{V}_1 \circ \mathfrak{V}_2 \circ \ldots \circ \mathfrak{V}_n$ *is carried by the iterated verbal wreath product* X_1 *defined (downward) inductively by*

$$X_n = F_{m_n}(\mathfrak{V}_n),$$
$$X_i = F_{m_i}(\mathfrak{V}_i)\ \mathrm{wr}_{\mathfrak{V}_i} X_{i+1}, \quad i \in \{1, \ldots, n-1\}$$

(*where, as in* (1.5.7), *we choose* $m_i = 0$ *if* $\mathfrak{V}_i = \mathfrak{E}$). (1.7.12)

Proof. Now the split-laws of $\mathfrak{V}_1 \circ \ldots \circ \mathfrak{V}_n$ are determined by $V_i(Y_i)$, $i = 1, \ldots, n$ since a split-group G belongs to $\mathfrak{V}_1 \circ \ldots \circ \mathfrak{V}_n$ if and only if it has these split-laws. The split-free split-group of rank $\boldsymbol{m}$ in $\mathfrak{V}_1 \circ \ldots \circ \mathfrak{V}_n$ is, by definition, $\boldsymbol{Q}(\boldsymbol{m})/S(\boldsymbol{Q}(\boldsymbol{m}))$ where

$$S = cl(\{V_1(Y_1), \ldots, V_n(Y_n)\}).$$

If $S_i = V_i(Y_i)$, then $S(\boldsymbol{Q}(\boldsymbol{m}))$ is the normal closure in $\boldsymbol{Q}(\boldsymbol{m})$ of all $S_i(\boldsymbol{Q}(\boldsymbol{m}))$. We construct $F_{\boldsymbol{m}}(\mathfrak{V}_1 \circ \ldots \circ \mathfrak{V}_n)$ by successively factoring out of $\boldsymbol{Q}(\boldsymbol{m})$, the normal closures of the $S_i(\boldsymbol{Q}(\boldsymbol{m}))$. Write $\boldsymbol{Q}(\boldsymbol{m}) = A_1 B_2$ in the usual notation: and at the first stage, since

$$A_1 = \Pi^*\{\hat{Y}_1^b : b \in B_2\}$$

(where $\hat{Y}_1 * \ldots * \hat{Y}_n$ is the carrier of $\boldsymbol{Q}(\boldsymbol{m})$), and $S_1(\boldsymbol{Q}(\boldsymbol{m}))^{\boldsymbol{Q}(\boldsymbol{m})} = V_1(A_1)$, we get

$$A_1/V_1(A_1) \cong (\Pi^*\{F_{m_1}(\mathfrak{V}_1) : b \in B_2\})/V_1(\Pi^*\{F_{m_1}(\mathfrak{V}_1) : b \in B_2\})$$
$$= \mathfrak{V}_1 \Pi\{F_{m_1}(\mathfrak{V}_1) : b \in B_2\}$$

(see 18.22, 18.23, 18.31 in Hanna Neumann (1967)). Hence

$$\boldsymbol{Q}(\boldsymbol{m})/S_1(\boldsymbol{Q}(\boldsymbol{m}))^{\boldsymbol{Q}(\boldsymbol{m})} = F_{m_1}(\mathfrak{V}_1)\ \mathrm{wr}_{\mathfrak{V}_1}(\Pi^*\{\hat{Y}_i : 2 \leqslant i \leqslant n\}).$$

Using theorem 1.4.12 and well-known properties of verbal wreath products, we arrive, by induction, at the assertion of the theorem.

THEOREM. *Let* $\mathscr{S}$ *be a subvariety of* $\mathfrak{W}\sigma$ *and let* $\{a_{ij} : 1 \leqslant i \leqslant n, j \in J\}$ *be a set of split-free generators for the split-free split-group* $\boldsymbol{F}_r(\mathscr{S})$ *of rank* $\boldsymbol{r} = (m_1, \ldots, m_n)$, *where* $m_i = r = |J|$ *if* y_{i1} *is not in* $\boldsymbol{S}$, *and* $m_i = 0$ *otherwise. Then the sub-*group *F of* $F_r(\mathscr{S})$ *generated by the elements*

$$b_j = a_{1j} a_{2j} \ldots a_{nj}, \quad j \in J,$$

(*if* $m_i = 0$ *put* $a_{ij} = 1$) *is free on* $\{b_j : j \in J\}$ *and is isomorphic to* $F_r(\mathscr{S}\tau)$. (1.7.13)

Proof. We verify first that if $\boldsymbol{G}$ is in $\mathscr{S}$ and if $\alpha : \{b_j : j \in J\} \to G$, then α can be extended to a homomorphism of F into G. Write, for each j in J,

$$b_j\alpha = a'_{1j} a'_{2j} \ldots a'_{nj}, \quad a'_{ij} \in A_i(\boldsymbol{G}),\ i \in \{1, \ldots, n\},$$

and define $\beta : \boldsymbol{F}_r(\mathscr{S}) \to \boldsymbol{G}$ by

$$a_{ij}\beta = a'_{ij}, \quad i \in \{1, \ldots, n\}, j \in J,$$

and (1.5.8). Then $\beta|F$ provides the extension of α we want. Finally note that if H is in $\mathscr{S}\tau$ then there exist $\boldsymbol{G}_i$ in $\mathscr{S}(i \in I)$ such that H is a factor K/N where K is a subgroup of $\Pi\{G_i : i \in I\}$, and if $\gamma : \{b_j : j \in J\} \to H$ is any mapping use what we have just proved to define $\delta : F \to K$ such that $(b_j \delta) N = b_j \gamma$ $(j \in J)$; then δ followed by the natural homomorphism from K onto H does what we want.

Corollary (cf. Šmel'kin 1965). *Let $F_r(\mathfrak{W})$, the free group of $\mathfrak{W}$ of rank r be freely generated by the set $\{f_j : j \in J, |J| = r\}$; and let $\boldsymbol{F}_r(\mathfrak{W}\sigma)$, the split-free split-group of $\mathfrak{V}_1 \circ \ldots \circ \mathfrak{V}_n$ of rank $\boldsymbol{r} = (m_1, \ldots, m_n)$ (where $m_i = r$ unless $\mathfrak{V}_i = \mathfrak{E}$ in which case $m_i = 0$), be freely generated by $\{a_{ij} : 1 \leqslant i \leqslant n, j \in J\}$. Then the mapping ν_r: $F_r(\mathfrak{W}) \to F_r(\mathfrak{W}\sigma)$ defined by*

$$f_j \nu_r = a_{1j} a_{2j} \ldots a_{nj}, \quad j \in J,$$

can be extended to a monomorphism. (1.7.14)

Proof. From (1.7.13) $F_r(\mathfrak{W})\nu_r$ is (isomorphic to) the free group of rank r of $\mathfrak{W}\sigma\tau$ on free generating set $\{f_j \nu_r : j \in J\}$. However, $\mathfrak{W}\sigma\tau = \mathfrak{W}$ by virtue of 22.32 in Hanna Neumann (1967). Hence the mapping $a_{1j} a_{2j} \ldots a_{nj} \to f_j$ $(j \in J)$ can be extended to a homomorphism; that is, ν_r is one-to-one.

Theorem. *The split-laws of $\mathfrak{U}\sigma$ are based on $\cup\{V_i(Y_i), U(Q_n) : 1 \leqslant i \leqslant n\}$; and $U(\boldsymbol{F}_r(\mathfrak{W}\sigma))$ carries $cl(U\nu_r)$.*

The laws of $\mathscr{S}\tau$ are based on those of $\mathfrak{W}$ together with $(S \cap F_\infty(\mathfrak{W})\nu_\infty)\nu_\infty^{-1}$. (1.7.15)

Proof. Now $\boldsymbol{F}_\infty(\mathfrak{W}\sigma)/U(\boldsymbol{F}_\infty(\mathfrak{W}\sigma))$ belongs to $\mathfrak{U}\sigma$ and $U(Q_n)$ is certainly contained in the split-laws of $\mathfrak{U}\sigma$; this takes care of the first assertion. That $U(\boldsymbol{F}_r(\mathfrak{W}\sigma)) = cl(U\nu_r)$ is proved by standard tricks which we here omit.

To prove the remaining statement, let $\mu_1 : F_\infty(\mathfrak{W}) \to F_\infty(\mathscr{S}\tau)$ and $\mu_2 : \boldsymbol{F}_\infty(\mathfrak{W}\sigma) \to \boldsymbol{F}_\infty(\mathscr{S})$ be the natural homomorphisms, and let $\nu : F_\infty(\mathscr{S}\tau) \to \boldsymbol{F}_\infty(\mathscr{S})$ be defined by

$$(f_j \mu_1)\nu = f_j \nu_\infty \mu_2, \quad j \in \{1, 2, \ldots\}.$$

Then (1.7.13) yields that ν extends to a monomorphism. Hence w in $F_\infty(\mathfrak{W})$ is a law in $\mathscr{S}\tau$ if and only if $w\mu_1 = 1$, that is if and only if $w\mu_1\nu = w\nu_\infty\mu_2 = 1$, that is if and only if $w\nu_\infty$ belongs to $S \cap F_\infty(\mathfrak{W})\nu_\infty$.

Chapter 2. Miscellaneous results

In this brief chapter we record some general results about lattices of split-varieties, and then introduce the bivarieties, with which the remainder of this paper is principally concerned.

2.1. *Lattices of split-varieties*

Theorem. *The split-varieties of the same species n form a modular lattice with respect to (the inclusion order and) the join and meet defined in* (1.5.12). (2.1.1)

Proof. By virtue of (1.5.4) and (1.5.13) it is sufficient to show that the normal, fully invariant sub-split-groups of $\boldsymbol{Q}_n$ form a modular lattice with respect to the inclusion order. This is clear, since if $\boldsymbol{S}$, $\boldsymbol{T}$ are normal and fully invariant in $\boldsymbol{Q}_n$, $\boldsymbol{S} \cap \boldsymbol{T}$ and $\boldsymbol{ST}$ are also, and therefore the normal, fully invariant sub-split-groups form a sublattice of the modular lattice of the normal subgroups of Q_n.

Because of this modularity, many results which are essentially lattice-theoretic can be taken over to our situation; all here are quoted without proof. The first is well known, particularly as a statement about varieties of groups.

Theorem. *If $\mathscr{S}$ is a split-variety which has a finite basis for its split-laws, then every sub-split-variety of $\mathscr{S}$ has a finite basis if and only if every descending chain of sub-split-varieties of $\mathscr{S}$ breaks off.* (2.1.2)

Of course, if there existed an infinite descending chain, $\mathfrak{V}_1 > \mathfrak{V}_2 > \ldots$ say, of varieties of groups, then we could trivially construct an infinite descending chain of split-varieties of arbitrary species ($\mathfrak{V}_1 \circ \mathscr{S} > \mathfrak{V}_2 \circ \mathscr{S} > \ldots$ where $\mathscr{S}$ is any split-variety).

The second result noted here I first proved for varieties of groups (see 16.25 in Hanna Neumann 1967). It is, however, a much older result about modular lattices, due to Pickert (1949).

Theorem. *If $\mathscr{S}$, $\mathscr{T}$ are split-varieties of the same species, each of which has descending chain condition on sub-split-varieties, then $\mathscr{S} \vee \mathscr{T}$ does also.* (2.1.3)

By entirely similar methods one also proves

Theorem. *A split-variety $\mathscr{S}$ has descending chain condition on sub-split-varieties if and only if there exists a subvariety $\mathscr{S}_0$ of $\mathscr{S}$ such that $\mathscr{S}_0$ has descending chain condition on sub-split-varieties, and also all descending chains between $\mathscr{S}$ and $\mathscr{S}_0$ break off.* (2.1.4)

2.2. *The bivariety $\mathfrak{A} \circ \mathfrak{A}$*

From now on we will be concerned almost exclusively with varieties of bigroups (bivarieties), mostly, indeed, with subvarieties of $\mathfrak{A} \circ \mathfrak{A}$. It is convenient to modify our notation to suit this situation. Thus we shall drop double subscripts and write $Y_m * Z_n$ for the carrier of the absolutely split-free bigroup of rank (m, n), with split-free generating set

$$\{y_i : i \in J_1, \ |J_1| = m\} \cup \{z_j : j \in J_2, \ |J_2| = n\}.$$

We now restate several results for the case of bivarieties, all of them special cases of theorem 1.4.5.

Theorem. *If q is a biword, then q is equivalent to a set $V_1 \cup V_2 \cup S$ of uniform biwords, where V_1, V_2 are contained in Y_∞, Z_∞ respectively, and where each element of S is a product of powers of left-normed commutators each of which has entries from both $\{y_i : i \in I^+\}$ and $\{z_j : j \in I^+\}$.* (2.2.1)

Proof. Put $V_1 = \{q_J : J \subseteq J_1\}$, $V_2 = \{q_J : J \subseteq J_2\}$ and $S = \{q_J : J \nsubseteq J_1, J \nsubseteq J_2\}$ using the notation of theorem 1.4.5, and these sets satisfy the assertion of the theorem.

COROLLARY. *Each subvariety of* $\mathfrak{A} \circ \mathfrak{B}$ *is determined by the bilaws of* $\mathfrak{A} \circ \mathfrak{B}$ *together with a set* $\{y_1^m\} \cup \hat{V} \cup S$ *of uniform biwords, where m is a non-negative integer,* $\hat{V}$ *is contained in* $Z_\infty - V$, *and where each element of S is a product of powers of commutators of the type*

$$[y_1, w_1, \ldots, w_r]$$

with each w_k *a left-normed commutator, not lying in* $cl(V \cup \hat{V})$, *whose entries are from* $\{z_j, z_j^{-1} : j \in I^+\}$. (2.2.2)

Proof. If $\mathscr{T}$ is a subvariety of $\mathfrak{A} \circ \mathfrak{B}$, then $\mathscr{T}$ is contained in $\mathfrak{A}_m \circ \mathfrak{B}'$ ($m \geqslant 0$, $\mathfrak{B}' \leqslant \mathfrak{B}$). Choose m minimal so that $\{y_1^m, [y_1, y_2]\}$ is a basis for the laws of var $\{A_1(G) : G \in \mathscr{T}\}$. If $\mathfrak{B}'$ is chosen minimally, write $\hat{V}$ for a set of uniform words which determine V' modulo V.

By (2.2.1) we are left to consider 'genuine' commutator biwords in T, call one t, say. Now t is equivalent to a set of uniform biwords t_J, by (1.4.5), and each t_J is a product of left-normed commutators, say $t_J = c_1 c_2 \ldots c_k$. Since $[y_1, y_2]$ is a bilaw in $\mathscr{T}$ we may assume that each c_j has one, and only one, entry from the set $\{y_i, y_i^{-1} : i \in I^+\}$ and since t_J is uniform, that each c_j has one and only one entry from the set $\{y_1, y_1^{-1}\}$. Modulo the bilaws of $\mathfrak{A} \circ \mathfrak{B}$ we then have

$$c_j = [y_1, w_1^{(j)}, \ldots, w_r^{(j)}]^{\pm 1}, \quad 1 \leqslant j \leqslant k$$

for some left-normed commutators $w_1^{(j)}, \ldots, w_r^{(j)}$. This completes the proof.

COROLLARY. *Every subvariety of* $\mathfrak{A} \circ \mathfrak{A}$ *is determined by the bilaws of* $\mathfrak{A} \circ \mathfrak{A}$ *together with a set* $\{y_1^m, z_1^n\} \cup S$ *of uniform biwords, where m, n are non-negative integers, and where every element s of S is a product of powers of commutators of the type*

$$[y_1, z_1^{\lambda_1}, \ldots, z_r^{\lambda_r}]$$

with r depending only on s, and $\lambda_1, \ldots, \lambda_r$ *all non-zero; if n is not zero then* $\lambda_j < n$, $j \in \{1, \ldots, r\}$. (2.2.3)

Proof. From (2.2.2) we have that every element of S can be written as a product of powers of commutators of the type $[y_1, z_{i1}^{\epsilon_1}, \ldots, z_{iu}^{\epsilon_u}]$, where $\epsilon_j = \pm 1$ and where $\{i_1, \ldots, i_u\} = \{1, \ldots, r\}$, say. If, for example, $i_1 = i_2$ then, since

$$[y_1, z_{i_1}^{\epsilon_1 + \epsilon_2}][y_1, z_{i_1}^{\epsilon_1}]^{-1}[y_1, z_{i_2}^{\epsilon_2}]^{-1} = [y_1, z_{i_1}^{\epsilon_1}, z_{i_2}^{\epsilon_2}]$$

we may replace this product by one of the desired type. That the z's can be re-arranged into increasing order of their subscripts follows since, modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$, y_1 is in the centralizer of the derived group of a metabelian group.

COROLLARY. *Every sub-bivariety of* $\mathfrak{A} \circ \mathfrak{A}$ *is determined by the bilaws of* $\mathfrak{A} \circ \mathfrak{A}$ *together with a set* $\{y_1^m, z_1^n\} \cup T$ *of uniform biwords, where m, n are non-negative integers and where every element of T is a product of powers of commutators of the type*

$$[y_1, \mu_1 z_1^{\epsilon_1}, \ldots, \mu_r z_r^{\epsilon_r}]$$

with $\mu_1, \ldots, \mu_r$ *natural numbers and* $\epsilon_1, \ldots, \epsilon_r = \pm 1$; *moreover, if* $n > 0$ *then* $\mu_1 < n$ *and* $\epsilon_i = 1$, $i \in \{1, \ldots, r\}$. (2.2.4)

Proof. Use (2.2.3) and the commutator identity (0.2.1).

Finally, in this chapter, a result of a completely different character. Note that the bivariety $\mathfrak{A} \circ \mathfrak{A}$ consists of bigroups which are metabelian *qua* groups. One of the nice features of such groups, from a varietal standpoint, is that finitely generated ones are residually finite, and therefore every subvariety of $\mathfrak{A}\mathfrak{A}$ is generated by finite groups. We implicitly adapt this very deep result of Philip Hall (1959) to our situation, in the next theorem.

THEOREM. *A bigroup* $\mathbf{G}$ *is residually finite* qua *bigroup if* G *is residually finite. Consequently every sub-bivariety of* $\mathfrak{A}\circ\mathfrak{A}$ *is generated by finite bigroups.* (2.2.5)

Proof. Let g be a non-trivial element of G. There exists a normal subgroup N of G not containing g such that $|G:N|$ is finite. Write

$$A_1(\mathbf{G})\cap N = A_1^*, \quad A_2(\mathbf{G})\cap N = A_2^*,$$

and then $|A_1(\mathbf{G}):A_1^*| = |A_1(\mathbf{G})\,N:N|$ and $|A_2(\mathbf{G}):A_2^*| = |A_2(\mathbf{G})\,N:N|$ are both finite. Hence

$$|G:A_1^*A_2^*| \leqslant |A_1(\mathbf{G}):A_1^*|\,.\,|A_2(\mathbf{G}):A_2^*|$$

is finite. Finally put $N^* = (A_1^*A_2^*)^G$, and then $A_1^*A_2^* \leqslant N^* \leqslant N$ so that N^* is normal, of finite index and avoids g; and as it carries a sub-bigroup of $\mathbf{G}$, the proof is complete.

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Chapter 3. Critical bigroups in $\mathfrak{A} \circ \mathfrak{A}$

In this chapter we define critical split-groups, by analogy with critical groups, (51.31 in Hanna Neumann 1967), deduce some elementary facts about them, and then turn our attention to the structure of certain critical bigroups in $\mathfrak{A} \circ \mathfrak{A}$.

3.1. *Critical split-groups*

Definition. *A finite split-group is* critical *if it is not in the split-variety generated by its proper sub-split-groups and proper quotient split-groups.* (3.1.1)

Clearly we have

Theorem. *If* $\mathbf{G}$ *is a split-group and* G *is a critical group, then* $\mathbf{G}$ *is critical.* (3.1.2)

Theorem. *A critical split-group has a unique minimal normal sub-split-group.* (3.1.3)

Proof. If not, then there exist non-trivial normal sub-split-groups $\mathbf{N}_1$, $\mathbf{N}_2$ of $\mathbf{G}$ with $\mathbf{N}_1 \cap \mathbf{N}_2 = 1$; and then $\mathbf{G}$ is a subdirect product of $\mathbf{G}/\mathbf{N}_1$, $\mathbf{G}/\mathbf{N}_2$.

An example of the situation in theorem 3.1.2 occurs when G is the symmetric group of permutations on three letters, $A_1(\mathbf{G})$ the normal 3-cycle and $A_2(\mathbf{G})$ a 2-cycle. However, the converse of (3.1.2) is not true: the carrier of a critical split-group need not be a critical group. An example of this is the bigroup $\mathbf{G}$ carried by the wreath product $G = C_p \operatorname{wr} (C_p \times C_p)$ in the natural way: $A_1(\mathbf{G})$ is the base group of G and $A_2(\mathbf{G})$ is the top group.

Clearly a split-group which is monolithic as a group has a unique, minimal normal sub-split-group. In certain cases the converse is true:

Lemma. *If* $\mathbf{G}$ *is a bigroup which has a unique minimal normal sub-bigroup, and* $A_1(\mathbf{G})$ *is abelian, then* G *is monolithic.* (3.1.4)

Proof. Suppose that N is a non-trivial normal subgroup of G. If $N \cap A_1(\mathbf{G})$ is not 1 then we are finished since $N \cap A_1(\mathbf{G})$ carries a normal sub-bigroup of $\mathbf{G}$. Hence suppose that $N \cap A_1(\mathbf{G}) = 1$; then as $A_1(\mathbf{G})$ is normal in G we have that N centralizes $A_1(\mathbf{G})$ and therefore that

$$C_G(A_1(\mathbf{G})) > A_1(\mathbf{G}).$$

It follows that $1 < C_G(A_1(\mathbf{G})) \cap A_2(\mathbf{G}) \trianglelefteq G$. Hence we have a contradiction unless $A_1(\mathbf{G}) = 1$, in which case the theorem is trivially true.

In the bivariety $\mathfrak{A} \circ \mathfrak{A}$ the conditions of (3.1.4) are certainly satisfied. In such cases we shall use 'monolithic' for brevity, and denote the monolith of G by σG. Note that the carrier of $\sigma \mathbf{G}$ is σG.

Lemma. *If a split-variety* $\mathscr{S}$ *is generated by finite split-groups then it is generated by critical split-groups.* (3.1.5)

Proof. Let $\mathscr{S}_0$ be the sub-split-variety of $\mathscr{S}$ generated by the critical split-groups in $\mathscr{S}$. If $\mathscr{S}_0$ is a proper subvariety of $\mathscr{S}$, then there exists a finite $\mathbf{G}$ in $\mathscr{S} - \mathscr{S}_0$, which we may suppose to have minimal order. Every proper sub-split-group and every proper quotient split-group of $\mathbf{G}$ then lies in $\mathscr{S}_0$, but $\mathbf{G}$ does not. This means that $\mathbf{G}$ is critical. We have thus produced a contradiction and hence $\mathscr{S}_0 = \mathscr{S}$.

Lemma (cf. theorem 4 in Powell 1964). *If* $\mathbf{G}$ *is a critical bigroup and* $A_1(\mathbf{G})$ *is abelian, then* $A_1(\mathbf{G})$ *contains a unique maximal normal subgroup of* G. (3.1.6)

Proof. If N_1, N_2 are maximal normal subgroups of G in $A_1(\mathbf{G})$, then N_1A_2, N_2A_2 carry sub-bigroups of $\mathbf{G}$ (writing $A_i = A_i(\mathbf{G})$, $i = 1, 2$). We shall show that $\mathbf{G} \in \operatorname{svar}\{N_1A_2, N_2A_2\}$. Suppose that q is a bilaw in both N_1A_2 and N_2A_2. Since $N_1N_2 = A_1$ and since $A_2(N_1A_2) = A_2$, we may suppose, by virtue of (2.2.2), that q is a product of commutators of the form

$$[y_1, w_1, \ldots, w_t]^{\pm 1}$$

for some words $w_1, \ldots, w_t$ in $A_2(\mathbf{Q}_2)$. Let $\alpha: \mathbf{Q}_2 \to \mathbf{G}$ be an arbitrary morphism. We write $y_1\alpha = a_1a_2$, $a_1 \in N_1$, $a_2 \in N_2$ (not necessarily uniquely). Define $\alpha_j: \mathbf{Q}_2 \to N_jA_2$, $j = 1, 2$, by

$$y_1\alpha_j = a_j, \quad z_i\alpha_j = z_i\alpha, \quad j = 1, 2, \quad i \in I^+.$$

Then
$$\begin{aligned}[y_1, w_1, \ldots, w_t]\alpha &= [y_1\alpha, w_1\alpha, \ldots, w_t\alpha] \\ &= [y_1\alpha_1, w_1\alpha_1, \ldots, w_t\alpha_1] \, . \, [y_1\alpha_2, w_1\alpha_2, \ldots, w_t\alpha_2].\end{aligned}$$

Hence $q\alpha = (q\alpha_1)(q\alpha_2) = 1$, showing that q is a bilaw in $\mathbf{G}$. This completes the proof.

Finally in this section an analogue of the well-known fact that critical groups which are nilpotent, are p-groups.

Theorem. *If $\mathbf{G}$ is a finite monolithic split-group and G is nilpotent, then for some prime p, G is a p-group.* (3.1.7)

Proof. If G is nilpotent and finite, its Sylow subgroups are fully invariant, hence carry normal sub-split-groups whose pair-wise intersections are trivial, so $\mathbf{G}$ cannot be monolithic unless G has only one Sylow subgroup.

3.2. *Non-nilpotent critical bigroups in $\mathfrak{A} \circ \mathfrak{A}$*

Throughout the remainder of this chapter $\mathbf{G} = (G, A, B)$ will be a critical, non-nilpotent bigroup contained in $\mathfrak{A} \circ \mathfrak{A}$; the notation introduced in theorem 3.2.1 will also be carried through.

Theorem. *If $\mathbf{G} = (G, A, B) \in \mathfrak{A} \circ \mathfrak{A}$ is critical and not nilpotent, then*

(i) *A is a p-group, for some prime p, it is self-centralizing in G, and is the derived group $G_{(2)} = G'$ of G.* If *$B = H \times K$ where H is the Sylow p-subgroup of B, then*

(ii) *$F = AH$ is the centralizer of the monolith σG of G, and F is the Fitting subgroup of G;*

(iii) *K is a p'-cycle which acts faithfully and irreducibly on σG.*

Moreover

(iv) *Every non-trivial element of K acts fixed point free on A,*

and

(v) *K acts faithfully and irreducibly on A/N,*

where

(vi) *$N = A^p[A, H]$ is the unique maximal G-normal subgroup of A.* (3.2.1)

Proof. Since $\mathbf{G}$ is critical it has a unique minimal normal sub-bigroup $\sigma\mathbf{G}$ whose carrier, by lemma 3.1.4, is the monolith σG of G.

If A were not a p-group, we could write it as a direct product of Sylow subgroups, each of which, being characteristic in A would be normal in G, contradicting the monolithicity of G; hence A is a p-group for some prime p. If A were not self-centralizing, then $A < C_G(A)$ would imply $1 < C_G(A) \cap B \trianglelefteq G$, again contradicting the monolithicity of G.

Since B is abelian, G' is contained in A; and since G is not nilpotent, there exists an integer t such that

$$1 \neq G_{(t)} = G_{(t+1)} = \ldots \leqslant A.$$

By a result of Schenkman (1955), G splits over $G_{(t)}$, say

$$G = G_{(t)}\, B_0, \quad G_{(t)} \cap B_0 = 1.$$

Therefore $A = G_{(t)}\,(A \cap B_0)$; but $A \cap B_0$ is normal in B_0 since A is normal in G, $A \cap B_0$ is normal in A since A is abelian: hence $A \cap B_0$ is normal in G, and so $A \cap B_0 = 1$ because G is monolithic and $A \cap B_0$ avoids $G_{(t)}$. That is,

$$A \leqslant G_{(t)} \leqslant G' \leqslant A,$$

or $G' = A$. This disposes of (i).

We can describe σG more exactly: if F has class c precisely, and if $F_{(c)}$ has exponent p^r, then

$$\sigma G = F_{(c)}^{p^{r-1}} = \{z \in Z(F) : z^p = 1\}. \tag{3.2.2}$$

For, $F_{(c)}^{p^{r-1}}$ is non-trivial and characteristic in F, therefore normal in G, and so contains σG. If this inclusion were proper then, by Maschke's theorem, σG would have a non-trivial, K-admissible complement in $F_{(c)}^{p^{r-1}}$ which, being in the centre of F, would be normal in G, a contradiction. A similar argument proves the remainder of (3.2.2).

The same argument can be used to prove that K acts irreducibly on σG. We shall now show that K acts not just faithfully on σG, but that every non-trivial element of K acts fixed point free on A. To this end suppose that there exists a non-trivial element k of K, and a non-trivial element x of A such that

$$x^k = x.$$

If we write

$$\hat{A} = \{a \in A : a^k = a\},$$

then $\hat{A}$ is a non-trivial normal subgroup of G in A and, by a well-known result of representation theory (for example, lemma, p. 455 in Higman 1956), $\hat{A}$ has a B-admissible complement A^* in A. But then A^* is normal in G since A is abelian and therefore $A^* = 1$ since G is monolithic; that is $\hat{A} = A$. In this case $\langle k \rangle$ is central in G, contradicting the existence of a monolith in G. It follows that a non-trivial element of K fixes no non-trivial element of A. Thus F is the centralizer of σG, K acts faithfully (and irreducibly) on σG and so K is cyclic, and F is the Fitting subgroup of G. This completes the proof of (ii), (iii) and (iv).

By lemma 3.1.6 there exists a unique maximal normal subgroup of G contained in A; call it N. Hence $A^p[A,H]$ is contained in N since both A^p and $[A,H] = F'$ are proper subgroups of A and both are normal in G. If the containment is proper, then $N/A^p[A,H]$ has a non-trivial K-admissible complement $T/A^p[A,H]$ say, in $A/A^p[A,H]$. But then T is normal in G and T is not contained in N, a contradiction to (3.1.6).

To finish the proof of the theorem we have to show that K acts faithfully on A/N, and to do this we use lemma 3.2.4 below (which will be useful later on as well): if a is in A, k in K and $[a,k]$ in N, then since N is characteristic in G, N admits the inverse of the automorphism α corresponding to k in (3.2.4) below; that is

$$a = [a,k]\,\alpha^{-1} \in N.$$

Hence K acts faithfully on A/N. The proof of theorem 3.2.1 is now complete.

Definition. *A bigroup* $\mathbf{G}_0 = (G_0, A_0, B_0)$ *in* $\mathfrak{A} \circ \mathfrak{A}$ *is* critical-like *if* $B_0 = H_0 \times K_0$ *where* K_0 *is cyclic and acts fixed-point-free on* A_0. (3.2.3)

LEMMA. *If G_0 is critical-like and k is a non-trivial element of K_0, then the mapping $\alpha: A_0 \to A_0$ defined by*

$$a\alpha = [a, k], \quad a \in A_0,$$

is an automorphism of A_0 which extends to an automorphism of G_0. (3.2.4)

Proof. Define α on the whole of G_0 by

$$(ba)\,\alpha = b[a, k], \quad a \in A_0, \quad b \in H_0 \times K_0.$$

This is an endomorphism since

$$((b_1 a_1)(b_2 a_2))\,\alpha = (b_1 b_2 a_1^{b_2} a_2)\,\alpha = b_1 b_2 [a_1^{b_2} a_2, k] = b_1 b_2 [a_1^{b_2}, k][a_2, k]$$
$$= b_1 b_2 [a_1, k]^{b_2} [a_2, k] = b_1 [a_1, k] \,.\, b_2 [a_2, k] = (b_1 a_1)\,\alpha \,.\, (b_2 a_2)\,\alpha;$$

and α is an automorphism since G_0 is finite and $b[a, k] = 1$ implies $b = 1$ and $[a, k] = 1$, which gives $a = 1$.

LEMMA. *If G_0 is critical-like, $|K_0| = t$, and $a_0, \ldots, a_{t-1}$ are elements of A_0 such that, for all k in K_0,*

$$\prod_{i=0}^{t-1} [a_i, ik] = 1, \qquad (3.2.5)$$

then $a_0 = \ldots = a_{t-1} = 1$. (3.2.6)

Proof. Put $k = 1$ and then $a_0 = 1$; we may suppose, therefore that the product is over the range $1 \leqslant i \leqslant t-1$. Let $K_0 = \langle k_0 \rangle$. Substitute k_0^j $(1 \leqslant j \leqslant t-1)$ for k in (3.2.5) in turn, and, using the terminology of (3.2.4) with α_j corresponding to k_0^j, we get

$$\prod_{i=1}^{t-1} a_i \alpha_j^i = 1, \quad 1 \leqslant j \leqslant t-1.$$

Working in the endomorphism ring of A_0 and utilizing the fact that $\alpha_i \alpha_j = \alpha_j \alpha_i$ $(1 \leqslant i, j \leqslant t-1)$ we deduce that

$$a_r \det(\alpha_j^i) = 1, \quad 1 \leqslant r \leqslant t-1.$$

Now $\det(\alpha_j^i)$ is the van der Monde determinant, and

$$\det(\alpha_j^i) = \left(\prod_{j=1}^{t-1} \alpha_j\right)\left(\prod_{u<v} (\alpha_u - \alpha_v)\right);$$

each α_j is an automorphism of A_0, and $\det(\alpha_j^i)$ will be an automorphism of A_0 if we can show that for $u < v$, $\alpha_u - \alpha_v$ is an automorphism of A_0: for $a \in A$,

$$a(\alpha_u - \alpha_v) = (a\alpha_u)(a\alpha_v)^{-1} = [a, k_0^u][a, k_0^v]^{-1}$$
$$= a^{-1} a^{k_0^u} \,.\, a^{-k_0^v} a = (a^{-1} a^{k_0^{v-u}})^{-k_0^u} = [a, k_0^{v-u}]^{-k_0^u}$$

and therefore $a(\alpha_u - \alpha_v) = 1$ implies $a = 1$. Hence $a_1 = \ldots = a_{t-1} = 1$ as asserted.

LEMMA. *Let G_0 be critical-like and $|K_0| = t$. If to each s-tuple $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s)$, where $0 \leqslant \mu_i \leqslant t-1$ $i \in \{1, \ldots, s\}$ there is an element $a(\boldsymbol{\mu})$ of A_0 such that for all $k_1, \ldots, k_s$ in K_0,*

$$\prod_{\boldsymbol{\mu}} [a(\boldsymbol{\mu}), \mu_1 k_1, \ldots, \mu_s k_s] = 1,$$

then $a(\boldsymbol{\mu}) = 1$ for all $\boldsymbol{\mu}$. (3.2.7)

Proof. For each ν in $\{0, \ldots, t-1\}$ write

$$a_\nu = \prod_{\nu = \mu_s} [a(\boldsymbol{\mu}), \mu_1 k_1, \ldots, \mu_{s-1} k_{s-1}];$$

then
$$\prod_{\nu=0}^{t-1} [a_\nu, \nu k_s] = 1$$

for all $k_s \in K_0$. Hence by (3.2.6), $a_0 = \ldots = a_{t-1} = 1$. We may now use induction to complete the proof.

3.3. *The criticality of G*

We aim to show in this section, that if **G** is as in (3.2.1), then G is a critical group. By lemma 3.1.4 and (1.2) of Kovács & Newman (1966) it suffices to show that G is not contained in the variety generated by its proper subgroups. To this end we calculate the maximal subgroups of G.

LEMMA. *If M is a maximal subgroup of G then either*
(*a*) $M = AHK_0$, *where K_0 is maximal in K;*
(*b*) $M = AH_0K$, *where H_0 is maximal in H,*
or (*c*) $M \cap F = NH$. (3.3.1)

Proof. Suppose that, as in (1.2.8), σ_1 is the retraction of G to B. Then if B properly contains $M\sigma_1$ we must have that M contains A for, if not, then $AM = G$ and therefore

$$B = G\sigma_1 = (AM)\sigma_1 = M\sigma_1.$$

Hence $M = A(M \cap B)$ and clearly $M \cap B$ must be maximal in B; that is, M has the form (*a*) or the form (*b*).

Assume, therefore, that $M\sigma_1 = B$; then $M \cap F = NH$. For, if M does not contain N, $G = NM$ and if $a \in A - N$,

$$a = xm, \quad x \in N, \quad m \in M, \tag{3.3.2}$$

and so $x^{-1}a = m \in (A-N) \cap M$. By virtue of (3.2.1) (vi), A is generated *qua* B-operator group by any element of $A - N$, and since $M\sigma_1 = B$ and A is abelian,

$$A = \langle m \rangle^B = \langle m \rangle^M \leqslant M.$$

In other words, $M = G$; hence M contains N. To finish off this case we show that if a is in $A - N$ and h in H, then ha is not an element of M. For, if k is non-trivial in K, there exists a' in A such that ka' is in M; and if ha belongs to M,

$$\begin{aligned}[ka', ha] &= [ka', a][ka', h][ka', h, a] \\ &= [k, a][a', h]\end{aligned}$$

belongs to M whence (as $[A, H] \leqslant N \leqslant M$), $[k, a]^{-1} = [a, k] \in M$. From (3.2.1) (v), $[a, k] \in (A - N) \cap M$, and an argument similar to that which disposed of (3.3.2) shows that $M = G$. Hence ha is not an element of M. It follows at once that $M\sigma_1 = B$ implies

$$M \cap F = NH,$$

as required in (*c*).

Note that not all the maximal subgroups of G are sub-bigroups. The ones which are not are those with $M \cap F = NH$ containing ka ($a \in A - N$, $k \in K$); in these cases, $M = \langle NH, ka \rangle$. A similar argument to the foregoing yields

LEMMA. *The maximal sub-bigroups of* **G** *are precisely FK_0, AH_0K and NHK, where H_0 is maximal in H, K_0 is maximal in K.* (3.3.3)

We are now ready to prove

THEOREM. *If* $\mathbf{G} = (G, A, B) \in \mathfrak{A} \circ \mathfrak{A}$ *is critical and not nilpotent, then G is a critical group.* (3.3.4)

Proof. Since **G** is critical, there exists a bilaw q of the maximal sub-bigroups of **G** which is not a bilaw in **G** itself. Because of the nature of the maximal sub-bigroups of **G**, q may be taken to be a genuine commutator biword, and using (2.2.3) we may assume q to take the form

$$q = \prod_{i=1}^{s} [y_1, z_1^{\alpha_{i1}}, \ldots, z_r^{\alpha_{ir}}]^{\epsilon_i},$$

where $\epsilon_i = \pm 1$, $\alpha_{ij} > 0$, $i \in \{1, \ldots, s\}$, $j \in \{1, \ldots, r\}$. Consider the word

$$w = \prod_{i=1}^{s} [x_1, x_2, x_3^{\alpha_{i1}}, \ldots, x_{r+2}^{\alpha_{ir}}]^{\epsilon_i}.$$

Then w is a law in every maximal subgroup of G, but not a law in G itself. For, if M is a maximal subgroup of G, then from (3.3.1) it follows that $(M'.M\sigma_1, M', M\sigma_1)$ is a proper sub-bigroup of $\mathbf{G}$; and each value of w in M is obtained by choosing arbitrary elements $m_1, \ldots, m_{r+2}$ of M and evaluating

$$\prod_{i=1}^{s} [m_1, m_2, m_3^{\alpha_{i1}}, \ldots, m_{r+2}^{\alpha_{ir}}]^{\epsilon_i} = \prod_{i=1}^{s} [m_1, m_2, (m_3\sigma_1)^{\alpha_{i1}}, \ldots, (m_{r+2}\sigma_1)^{\alpha_{ir}}]^{\epsilon_i};$$

this is clearly a value of q in a proper sub-bigroup, and is therefore 1. Hence w is a law in M.

On the other hand, since q is not a bilaw in $\mathbf{G}$, there exist elements a of A, $b_1, \ldots, b_r$ of B such that

$$\prod_{i=1}^{s} [a, b_1^{\alpha_{i1}}, \ldots, b_r^{\alpha_{ir}}]^{\epsilon_i} \neq 1.$$

From (3.2.4), if k is a non-trivial element of K, there exists a' in A with $a = [a', k]$; it follows that

$$\prod_{i=1}^{s} [a', k, b_1^{\alpha_{i1}}, \ldots, b_r^{\alpha_{ir}}]^{\epsilon_i}$$

is different from 1 and therefore that w is not a law in G. By the remark at the beginning of this section, G is critical. We shall see later that this theorem has a strong converse.

3.4. *The bigroup* $\mathbf{F}^*$

In this section we show that, in a sense, the bivariety generated by the critical bigroup $\mathbf{G}$ is determined by the bivariety generated by a certain sub-bigroup of $\mathbf{G}$ which sometimes turns out to be a little more manageable.

Recall that (3.2.1) (vi) ensures that if a_0 is an element of $A - N$, then A is generated, *qua* B-operator group, by a_0. Suppose that one such a_0 is chosen and fixed from now on. Write $A_0 = \langle a_0 \rangle^H$, $F_0 = A_0 H$ and
$$\mathbf{F}^* = \mathbf{F}_0 = (F_0, A_0, H).$$
This definition depends on a_0 but is unambiguous up to isomorphism, as the following result shows.

LEMMA. *If a_0, a_1 belong to $A - N$, then the mapping $a_0 \to a_1$ can be extended to an isomorphism of the corresponding sub-bigroups $\mathbf{F}_0$ and $\mathbf{F}_1$.* (3.4.1)

Proof. Suppose that $r = r(a_0, h_1, \ldots, h_t) = 1$ is a relation among the generating set $\{a_0\} \cup H$ for F_0. Every relation in H is a relation in both F_0 and F_1, so we may assume that r takes the form

$$r = \prod_{i=1}^{t} a_0^{\alpha_i h_i} = 1$$

for some integers α_i. Now there exist $b_1, \ldots, b_u$ in B such that

$$a_1 = \prod_{i=1}^{u} a_0^{\beta_i b_i}$$

for some integers β_i. Therefore

$$\begin{aligned} r(a_1, h_1, \ldots, h_t) &= \prod_{i=1}^{t} \left\{ \prod_{j=1}^{u} a_0^{\beta_j b_j} \right\}^{\alpha_i h_i} \\ &= \prod_{j=1}^{u} \left\{ \prod_{i=1}^{t} a_0^{\alpha_i h_i} \right\}^{\beta_j b_j}, \quad \text{since } A, B \text{ are abelian,} \\ &= 1. \end{aligned}$$

Hence, by von Dyck's theorem, the mapping $a_0 \to a_1$ and the identity mapping of H extend to a morphism $\boldsymbol{F}_0 \to \boldsymbol{F}_1$. Similarly, the mapping $a_1 \to a_0$ and the identity mapping of H extend to a morphism $\boldsymbol{F}_1 \to \boldsymbol{F}_0$. Consequently each is an isomorphism.

LEMMA. $\boldsymbol{F}$ and $\boldsymbol{F}^*$ *generate the same bivariety.* (3.4.2)

The proof of this is similar to that of (3.1.6), and we omit it.

It would be pleasant if it turned out that $\boldsymbol{F}^*$ was a critical bigroup. However, this is not in general the case. The best that can be said is (3.4.3) below. The trouble comes from the fact that $\boldsymbol{F}^*$ need not be monolithic; for example, there exists a non-nilpotent critical $\boldsymbol{G}$ in which $\boldsymbol{F}^*$ is carried by the central factor group of $C_4 \operatorname{wr} C_2$: this topic will be taken up again in (3.4.7).

LEMMA. *If* $\boldsymbol{G}$ *is as in* (3.2.1), *then* $\boldsymbol{F}^*$ *is not in the bivariety generated by its proper sub-bigroups.* (3.4.3)

This will follow from the next lemma, which is much more important from our point of view in the next two chapters.

LEMMA. *Let q be a commutator biword and t a positive integer. There exist biwords $q_1, \ldots, q_v$, depending on q, t, such that if q is a bilaw in a critical-like bigroup with $|K_0| = t$, then $q_1, \ldots, q_v$ are bilaws in* $(A_0 H_0, A_0, H_0) = \boldsymbol{F}_0$.

Conversely if $(G_1, A_1, H_1 \times K_1)$ is in $\mathfrak{A} \circ \mathfrak{A}$, $\exp K_1$ divides t, and $q_1, \ldots, q_v$ are bilaws in

$$(A_1 H_1, A_1, H_1) = \boldsymbol{F}_1,$$

then q is a bilaw in $\boldsymbol{G}_1$. (3.4.4)

Proof. Using (2.2.4) we may assume

$$q = \prod_{i=1}^{s} [y_1, \mu_{i1} z_1^{\epsilon_{i1}}, \ldots, \mu_{ir} z_r^{\epsilon_{ir}}]^{\alpha_i},$$

where μ_{ij} are all natural numbers, and $\epsilon_{ij} = \pm 1$. Suppose that q is a bilaw of the critical-like bigroup $\boldsymbol{G}_0$. Consider the biword

$$q^* = \prod_{i=1}^{s} [y_1, \mu_{i1} z_{1+r}^{\epsilon_{i1}} z_{1+2r}^{\epsilon_{i1}}, \ldots, \mu_{ir} z_{2r}^{\epsilon_{ir}} z_{3r}^{\epsilon_{ir}}]^{\alpha_i}.$$

In this expression for q^* expand each commutator, using repeatedly the identity (0.2.2) modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$. We get a product of powers of commutators each of which has y_1 as first entry and some $z_j^{\pm 1}$, $j \in \{r+1, \ldots, 3r\}$ in each other entry. Working modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$ we can collect to the front of each commutator all $z_j^{\pm 1}$ with $j \in \{r+1, \ldots, 2r\}$. Hence there exist biwords $q_1^*, \ldots, q_u^*$ such that, modulo $\mathfrak{A} \circ \mathfrak{A}(\boldsymbol{Q}_2)$,

$$q^* = \prod_{i=1}^{u} [q_i^*, \lambda_{i1} z_{1+2r}^{\eta_{i1}}, \ldots, \lambda_{ir} z_{3r}^{\eta_{ir}}],$$

where $q_1^*, \ldots, q_u^*$ are biwords which are products of powers of commutators each of which has as entries, y_1 in the first place, and $z_j^{\pm 1}$, $j \in \{r+1, \ldots, 2r\}$ in the other places; and where $\eta_{ij} = \pm 1$, $i \in \{1, \ldots, u\}$, $j \in \{2r+1, \ldots, 3r\}$.

Now consider

$$q^{**} = \prod_{i=1}^{u} [q_i^*, \lambda_{i1} z_{1+2r}^{\zeta_{i1}}, \ldots, \lambda_{ir} z_{3r}^{\zeta_{ir}}],$$

where $\zeta_{ij} = \eta_{ij}$ if $\eta_{ij} = 1$, and $\zeta_{ij} = t-1$ if $\eta_{ij} = -1$. Making repeated use of the identity (0.2.1) we can write, again modulo $\mathfrak{A} \circ \mathfrak{A}(\boldsymbol{Q}_2)$,

$$q^{**} = \prod_{i=1}^{v} [q_i, \nu_{i1} z_{1+2r}, \ldots, \nu_{ir} z_{3r}] \,.\, q',$$

where each q_i is a linear combination of q_j^*'s, where $0 \leqslant \nu_{ij} \leqslant t-1$ all i, j, and where q' is a (possibly empty) product of powers of commutators in each of which at least one of $z_{1+2r}, \ldots, z_{3r}$ occurs raised to a power which is a multiple of t.

Now suppose that $\alpha: \boldsymbol{Q}_2 \to \boldsymbol{F}_0$ is arbitrary and, for the moment, fixed. With each choice $k_1, \ldots, k_r$ in K_0, and α, associate a morphism $\beta: \boldsymbol{Q}_2 \to \boldsymbol{G}_0$ such that

$$y_i\beta = y_i\alpha, \quad i \in I^+,$$

$$z_{j+r}\beta = z_j\alpha, \quad z_{j+2r}\beta = k_j, \quad j \in \{1, \ldots, r\}.$$

Then if $\beta^*: \boldsymbol{Q}_2 \to \boldsymbol{G}_0$ is such that

$$y_i\beta^* = y_i\alpha, \quad i \in I^+,$$

$$z_j\beta^* = (z_j\alpha) . k_j, \quad j \in \{1, \ldots, r\},$$

we have

$$1 = q\beta^* = q^*\beta = q^{**}\beta = \prod_{i=1}^{v} [q_i\alpha, \nu_{i1}k_1, \ldots, \nu_{ir}k_r],$$

and this for all such β. Hence, by (3.2.7), $q_i\alpha = 1, i \in \{1, \ldots, v\}$, and since α was arbitrary, $q_1, \ldots, q_v$ are bilaws in $\boldsymbol{F}_0$.

Conversely, suppose that $q_1, \ldots, q_v$ are bilaws in (A_1H_1, A_1, H_1). Then if $\beta^*: \boldsymbol{Q}_2 \to \boldsymbol{G}_1$ is any morphism we can construct $\alpha: \boldsymbol{Q}_2 \to \boldsymbol{F}_1$ and $\beta: \boldsymbol{Q}_2 \to \boldsymbol{G}_1$ reversing the procedure in the foregoing proof. Then, so long as $\exp K_1$ divides t, we have $q_1\alpha = \ldots = q_v\alpha = 1$ implies $q\beta^* = 1$ and so q is a bilaw in $\boldsymbol{G}_1$.

Remark. The argument above is, of course, essentially a trigroup argument. However, it seems easier to treat it as we have done, than to develop the necessary conventions and terminology involved in considering $\boldsymbol{G}_0$ as a trigroup. (3.4.5)

Proof of (3.4.3). Since $\boldsymbol{G}$ is critical, there is a (commutator) biword q which is a bilaw in every maximal sub-bigroup of $\boldsymbol{G}$, but not in $\boldsymbol{G}$ itself. In particular q is a bilaw in the maximal sub-bigroups whose carriers are of the type

$$AH_0K, \quad NHK, \quad H_0 \text{ maximal in } H;$$

and these sub-groups are critical-like. It follows, therefore, that if $q_1, \ldots, q_v$ correspond to q by (3.4.4), then $q_1, \ldots, q_v$ are bilaws in all (AH_0, A, H_0) and in (NH, N, H). However $q_1, \ldots, q_v$ cannot all be bilaws in $\boldsymbol{F}$ since q is not a bilaw in $\boldsymbol{G}$. It remains to remark that the maximal sub-bigroups of $\boldsymbol{F}^*$ are precisely those carried by $AH_0 \cap F^*$ and $N_0H = NH \cap F^*$, by an argument similar to that of (3.3.1), and that they generate the same bivarieties as their counter-parts in $\boldsymbol{F}$.

We now prove a partial converse of (3.4.3). First note that, by (3.4.6) below, $\boldsymbol{F}^*$ is critical if it is monolithic. In (3.4.7) we show that critical bigroups $\boldsymbol{G}$ with prescribed, critical $\boldsymbol{F}^*$ and prescribed $|K|$ always exist, are unique, and that as far as generation of sub-bivarieties of $\mathfrak{A} \circ \mathfrak{A}$ is concerned, no others are necessary.

Theorem. *If* $\boldsymbol{G}$ *belongs to* $\mathfrak{A} \circ \mathfrak{A}$, *is monolithic, and not in the bivariety generated by its proper sub-bigroups, then* $\boldsymbol{G}$ *is critical.* (3.4.6)

Proof. Let $\boldsymbol{G} = (G, A, B)$. It follows as in (3.1.6) that there exists a unique maximal normal subgroup N of G contained in A, and hence that $N = A^p[A, H]$ where A is a p-group and H is the Sylow p-subgroup of B. Also it is easy to see that the maximal sub-bigroups of $\boldsymbol{G}$ are carried by AB_0, NB where B_0 is maximal in B. We show that $\boldsymbol{G}/\sigma\boldsymbol{G} \in \operatorname{svar}\{NB\}$. If q is a bilaw in NB we may assume it to be uniform, involving $y_1, z_1, \ldots, z_{t-1}$ $(t \geqslant 1)$. If $\alpha: \boldsymbol{Q}_2 \to \boldsymbol{G}$, define $\beta: \boldsymbol{Q}_2 \to NB$ by

$$y_1\beta = (y_1\alpha)^p, \quad z_i\beta = z_i\alpha, \quad i \in I^+,$$

and $\gamma: \boldsymbol{Q}_2 \to NB$ by

$$y_1\gamma = [y_1, z_t^r], \quad z_i\gamma = z_i\alpha, \quad i \in I^+,$$

where $r = \exp B/H$. It is easily seen that $(q\alpha)^p = q\beta = 1$, $[q, z_t^r]\alpha = q\gamma = 1$. Thus $q(\boldsymbol{G})$ lies in the socle of AH, that is, in $\sigma\boldsymbol{G}$. Hence q is a law in $\boldsymbol{G}/\sigma\boldsymbol{G}$. Since all proper quotient bigroups of $\boldsymbol{G}$ are quotient bigroups of $\boldsymbol{G}/\sigma\boldsymbol{G}$ it follows from the hypotheses that $\boldsymbol{G}$ is critical.

Theorem. *If $\boldsymbol{P}$ in $\mathfrak{A}\circ\mathfrak{A}$ is nilpotent and critical, with $A_1(\boldsymbol{P})$ non-trivial, then there exists to each natural number t which is prime to the order of P, a unique non-nilpotent critical bigroup $\boldsymbol{G}$ in $\mathfrak{A}\circ\mathfrak{A}$ with $|K| = t$ and $\boldsymbol{F}^* \cong \boldsymbol{P}$.*

If $(\hat{G}, \hat{A}, \hat{H} \times \hat{K})$ in $\mathfrak{A}\circ\mathfrak{A}$ is critical-like, with $\hat{A}\hat{H}$ a p-group and $\hat{A}$ non-trivial and self-centralizing, then there exist critical bigroups $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w$ such that each $\boldsymbol{F}_i^*$ is critical, each $|K_i| = |\hat{K}|$, and $\operatorname{svar}\{\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w\} = \operatorname{svar}\{\hat{\boldsymbol{G}}\}$. (3.4.7)

Proof. From (3.1.6), $A_1(\boldsymbol{P})$ is monogenic *qua* P operator group; also $\boldsymbol{P}$ is monolithic. Choose the natural number s so that t divides $p^s - 1$ but not $p^u - 1$ if $u < s$. Let $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_s$ be isomorphic copies of $\boldsymbol{P}$, say $\lambda_i: \boldsymbol{P}_1 \to \boldsymbol{P}_i$ is an isomorphism. If a_i in $A_1(\boldsymbol{P}_i)$ is such that

$$\langle a_i \rangle^{A_2(\boldsymbol{P}_i)} = A_1(\boldsymbol{P}_i),$$

we may suppose $a_i = a_1\lambda_i$, $i = \{1, \ldots, s\}$.

In the direct product $\boldsymbol{P}_1 \times \ldots \times \boldsymbol{P}_s$ write $A = A_1(\boldsymbol{P}_1 \times \ldots \times \boldsymbol{P}_s)$, and H for the diagonal of $A_2(\boldsymbol{P}_1 \times \ldots \times \boldsymbol{P}_s)$; that is

$$H = \{f : f(i) = f(1)\,\lambda_i \in A_2(\boldsymbol{P}_i)\}.$$

and set $\boldsymbol{F} = (AH, A, H)$. We aim to extend $\boldsymbol{F}$ by a t-cycle so that the resulting bigroup is critical.

Put $A_0 = \langle a_1, \ldots, a_s \rangle$ and let $K = \langle k : k^t = 1 \rangle$ be a cycle of order t. According to Cossey (1966, theorem 4.2.2) there exists a unique critical group A_0K; in this group let k induce an automorphism α on A_0. Define the action of α on H to be the identity mapping of H. Then α extends to an automorphism of $\boldsymbol{F}$. For, let

$$r = r(a_1, \ldots, a_s, h_1, \ldots, h_u) = 1$$

be a relation among the generating set $\{a_1, \ldots, a_s\} \cup H$ of F. Clearly $r = 1$ is equivalent to a set of relations

$$r_i = r_i(a_i, h_1, \ldots, h_u) = 1, \quad i \in \{1, \ldots, s\}.$$

Because of the way we have constructed F, $r_i = 1$ is a relation in F if and only if $r_i(a_j, h_1, \ldots, h_u) = 1$ is a relation in F, $i, j \in \{1, \ldots, s\}$. If $a_0 = \Pi_{i=1}^s a_i^{\beta_i}$ is any element of A_0, then

$$r_i(a_0, \ldots, h_u) = \prod_{j=1}^{s} r_i(a_j^{\beta_j}, h_1, \ldots, h_u)$$

$$= \prod_{j=1}^{s} r_i(a_j, h_1, \ldots, h_u)^{\beta_j} = 1.$$

By von Dyck's theorem, α may be extended to an endomorphism of F. Since $A_0\alpha = A_0$, $F\alpha = F$, and consequently α is an automorphism of $\boldsymbol{F}$.

Next we verify that (FK, A, HK) is critical. As a first step we show that K acts fixed point free on A. If $N = A^p[A, H]$, $N_i = A_1(\boldsymbol{P}_i)^p[A_1(\boldsymbol{P}_i), A_2(\boldsymbol{P}_i)]$ then $N_i = N \cap A_1(\boldsymbol{P}_i)$ and

$$N = N_1 \times \ldots \times N_s,$$

so that $A/N \cong A_1/N_1 \times \ldots \times A_s/N_s \cong A_0/A_0^p$ where the isomorphisms are K-isomorphisms. Hence K acts faithfully and irreducibly on A/N. Now there exist elements $h_1, \ldots, h_u$ of H and a non-negative integer γ such that

$$1 \neq x_i = [a_i, h_1, \ldots, h_u]^{p^\gamma} \in \sigma \boldsymbol{P}_i, \quad i \in \{1, \ldots, s\};$$

and the mapping $a_i N \to x_i$ extends to a K-homomorphism μ of A/N into $\sigma \boldsymbol{F}$, the socle of $\boldsymbol{F}$. In fact μ is a K-isomorphism since K acts faithfully and irreducibly on A/N and since clearly $\langle x_1, \ldots, x_s \rangle = \sigma F$. It follows that K acts faithfully and irreducibly on σF, and therefore fixed point free on A. Finally a calculation similar to that in the proof of (3.3.1) shows that the maximal sub-bigroups of FK are precisely those carried by AH_0K, AHK_0, NHK where H_0, K_0 are maximal in H, K respectively; and, as in the proof of (3.1.6), svar $(AH_0, A, H_0) =$ svar $(A_1(\boldsymbol{P}_1)\, H_0, A_1(\boldsymbol{P}), H_0)$ and also svar $(NH, N, H) =$ svar $(N_1 H, N_1, H)$. By hypothesis therefore, there exists a biword q which is a bilaw in the bigroups carried by AH_0, NH, but not in that carried by AH. If q involves the variables $z_1, z_2, \ldots, z_u$ from $\{z_1, z_2, \ldots,\}$ and $t_1, \ldots, t_v$ are the maximal divisors of t, not equal to 1 (if any), consider the biword

$$q'' = [q', z_{u+1}^{t_1}, \ldots, z_{u+v}^{t_v}],$$

where q' is obtained from q by replacing each z_i by z_i^t. Then q'' is a bilaw in all maximal sub-bigroups of FK but not in FK itself. Since FK is monolithic, (3.4.6) concludes the existence part of the proof.

In order to prove the uniqueness of $\boldsymbol{G}$, suppose that $\boldsymbol{G}_1$ is non-nilpotent and critical with $\boldsymbol{F}_1^* \cong \boldsymbol{F}^*$ and $K_1 \cong K$. Let a_0' belong to $A_1(\boldsymbol{F}_1) - N_1$. Then $A_0K = \langle a_0, K \rangle$ and $A_0'K = \langle a_0', K_1 \rangle$ are critical groups, and the result of Cossey (1966) mentioned earlier ensures that there exists an isomorphism $\alpha : A_0K \to A_0'K_1$ which takes A_0 to A_0' and K to K_1. It is easy to verify (for example as in (3.4.1)) that $\alpha | A_0$ extends to a morphism of $\boldsymbol{F}$ onto $\boldsymbol{F}_1$ which commutes with the action of K, and is one-to-one when restricted to $\boldsymbol{F}^*$. Hence there exists a morphism of $\boldsymbol{G}$ onto $\boldsymbol{G}_1$, the kernel of which, if not trivial, must contain $\sigma \boldsymbol{G}$, and therefore $\sigma \boldsymbol{F}^*$, a contradiction. Therefore $\boldsymbol{G}$ and $\boldsymbol{G}_1$ are isomorphic.

To prove the second assertion let $\hat{\boldsymbol{G}}$ be as stated. Now $\hat{\boldsymbol{F}} = (\hat{A}\hat{H}, \hat{A}, \hat{H})$ is contained in the bivariety irredundantly generated by some of its critical factors $\boldsymbol{F}_1^*, \ldots, \boldsymbol{F}_w^*$ say. We may suppose $A_1(\boldsymbol{F}_i^*)$ are all non-trivial. For, if $A_1(\boldsymbol{F}_1^*)$, say, were 1, then $\exp A_2(\boldsymbol{F}_1^*) > \exp A_2(\boldsymbol{F}_i^*)$, $i \in \{2, \ldots, w\}$ (or else $\boldsymbol{F}_1^*$ would be redundant), and then $\boldsymbol{F}_1^*, \ldots, \boldsymbol{F}_w^*$, and therefore $\hat{\boldsymbol{F}}$ would have a bilaw $[y_1, z_1^{p^\beta}]$ where $z_1^{p^\beta}$ is not a bilaw in $\hat{\boldsymbol{F}}$. But $A_1(\hat{\boldsymbol{F}})$ is non-trivial and self-centralizing in $\boldsymbol{F}$ and therefore we would have a contradiction. According to the first part of the theorem, we may construct critical bigroups $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w$ from $\boldsymbol{F}_1^*, \ldots, \boldsymbol{F}_w^*$ respectively, and the same cycle isomorphic to K. Then svar $\hat{\boldsymbol{G}} =$ svar $\{\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w\}$. For, if q is a commutator biword, and $q_1, \ldots, q_v$ correspond to q, t by lemma 3.4.4, then q is a bilaw in $\hat{\boldsymbol{G}}$ if and only if $q_1, \ldots, q_v$ are bilaws in $\hat{\boldsymbol{F}}$, hence if and only if $q_1, \ldots, q_v$ are bilaws in $\boldsymbol{F}_1^*, \ldots, \boldsymbol{F}_w^*$, and therefore if and only if q is a bilaw in $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w$; and biwords y_1^r, z_1^s are bilaws in $\hat{\boldsymbol{G}}$ if and only if they are bilaws in $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w$, by an easy argument which we omit.

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Chapter 4. Join-irreducibles in $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$

It was mentioned in the introduction that a classification of the join irreducibles in $\Lambda(\mathfrak{A} \circ \mathfrak{A})$ is one of the aims of this paper. To this end we restrict our attention in this chapter to the finite exponent bivarieties. As a by-product of the proofs here we get that the lattice $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ has minimum condition, and certain reduction theorems relating to questions of distributivity. However, the main results are theorem 4.1.16, where we determine all join-irreducibles in $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ in terms of join-irreducibles of prime-power exponent, and theorem 4.2.30, where we deal with the non-nilpotent ones of prime-power exponent. The only cases in which a complete classification of all subvarieties of a given bivariety is obtainable is for $\mathfrak{A}_m \circ \mathfrak{A}_n$, when m is nearly prime to n (this is done in § 4.3), and for subvarieties of $\mathfrak{A}_m \circ \mathfrak{A}_n$ in which the class of nilpotent bigroups is suitably restricted (§ 4.4).

4.1. *Subvarieties of $\mathfrak{A}_m \circ \mathfrak{A}_n$: reduction to the case of prime-power exponent*

We begin with a few remarks of a general character.

Definition. *If $\mathscr{B}$ is a bivariety, define*

$$\mathscr{B}\phi = \operatorname{svar}\{\mathbf{G} \in \mathscr{B} : \mathbf{G} \text{ critical}, A_1(\mathbf{G}) \neq 1\},$$
$$\mathscr{B}\psi = \{\mathbf{G} \in \mathscr{B} : A_1(\mathbf{G}) = 1\}.$$

Also define

$$\left.\begin{aligned} \Phi(\mathscr{B}) &= \{\mathscr{C}\phi : \mathscr{C} \leqslant \mathscr{B}\}, \\ \Psi(\mathscr{B}) &= \{\mathscr{C}\psi : \mathscr{C} \leqslant \mathscr{B}\}. \end{aligned}\right\} \tag{4.1.1}$$

Definition. *Denote the lattice of sub-split-varieties of a split-variety $\mathscr{S}$ by $\Lambda(\mathscr{S})$.* (4.1.2)

Lemma. *Each of $\Phi(\mathscr{B})$, $\Psi(\mathscr{B})$, equipped with the inclusion order inherited from $\Lambda(\mathscr{B})$, is a complete lattice. The mappings $\phi : \Lambda(\mathscr{B}) \to \Phi(\mathscr{B})$, $\psi : \Lambda(\mathscr{B}) \to \Psi(\mathscr{B})$ are onto lattice-homomorphisms.* (4.1.3)

Proof. Now $\Psi(\mathscr{B})$ is clearly a sublattice of $\mathscr{B}$, in fact equal to $\Lambda(\mathscr{B} \wedge \mathfrak{E} \circ \mathfrak{O})$ (where $\mathfrak{O}$ is the variety of all groups). In $\Phi(\mathscr{B})$, the join of any subset is equal to its join in $\Lambda(\mathscr{B})$, and the meet of any subset is the largest element of $\Phi(\mathscr{B})$ contained in all elements of the subset: indeed if $\mathscr{C}_i \leqslant \mathscr{B}$ $(i \in I)$, then

$$\dot{\wedge}\{\mathscr{C}_i\phi : i \in I\} = (\wedge\{\mathscr{C}_i : i \in I\})\,\phi.$$

(An instance of $\mathscr{C}_1 \dot{\wedge} \mathscr{C}_2 \neq \mathscr{C}_1 \wedge \mathscr{C}_2$ occurs in the lattice $\Lambda(\mathfrak{A}_4 \circ \mathfrak{A}_4 \wedge \mathfrak{N}_3)$ in § 4.4 with

$$\mathscr{C}_1 = \mathfrak{A}_2 \circ \mathfrak{A}_4 \wedge \mathfrak{N}_3,\ \mathscr{C}_2 = \mathscr{V}_3.)$$

That ψ is a homomorphism follows since the bilaws defining $\mathscr{C}\psi$ for any $\mathscr{C}$ are precisely $C \cap A_2(\mathbf{Q}_2) = C\sigma_1$ (by (1.2.8)), and σ_1 is a lattice homomorphism. To show that ϕ is a homomorphism we need the following lemma.

Lemma. *If $\mathbf{G}$ is critical with $A_1(\mathbf{G})$ non-trivial, and if*

$$\mathbf{G} \in \operatorname{svar}\{\mathbf{G}_j : j \in J\} \vee \mathfrak{E} \circ \mathfrak{O}$$

where, for each j, $A_1(\mathbf{G}_j)$ is non-trivial, then

$$\mathbf{G} \in \operatorname{svar}\{\mathbf{G}_j : j \in J\}. \tag{4.1.4}$$

Proof. If q is a bilaw in all $\mathbf{G}_j$ we may assume by virtue of (2.2.1) that either q is in $A_1(\mathbf{Q}_2)$ or q is in $A_2(\mathbf{Q}_2)$. Write $q' = q$ in the first case, and $q' = [y_1, q]$ in the second; then q' is a bilaw in all $\boldsymbol{G}_j$ and in $\mathfrak{E} \circ \mathfrak{D}$, whence in $\boldsymbol{G}$. Since $A_1(\boldsymbol{G})$ is non-trivial, and the centralizer of $A_1(\boldsymbol{G})$ in $A_2(\boldsymbol{G})$ is trivial, we deduce that q is a bilaw in $\boldsymbol{G}$. This completes the proof.

Returning to the proof of (4.1.3) we note that, if $\boldsymbol{G}$ in $\mathscr{C} \vee \mathscr{D}$ is critical, and $A_1(\boldsymbol{G})$ is not 1, then by (4.1.4), $\boldsymbol{G}$ belongs to $\mathscr{C}\phi \vee \mathscr{D}\phi$, whence

$$(\mathscr{C} \vee \mathscr{D})\,\phi \leqslant \mathscr{C}\phi \vee \mathscr{D}\phi.$$

As the converse inclusion is obvious this shows that ϕ is a join-homomorphism. By definition, ϕ is a meet homomorphism, so (4.1.3) is proved.

THEOREM. *If $\mathscr{B}$ is a bivariety in which every sub-bivariety is generated by finite bigroups, then $\Lambda(\mathscr{B})$ is a sub-direct product of $\Phi(\mathscr{B})$ and $\Psi(\mathscr{B})$.* (4.1.5)

Proof. In this case, if $\mathscr{C} \leqslant \mathscr{B}$, then

$$\mathscr{C} = \mathscr{C}\phi \vee \mathscr{C}\psi;$$

and therefore $\mathscr{C}\phi = \mathscr{D}\phi$, $\mathscr{C}\psi = \mathscr{D}\psi$ implies $\mathscr{C} = \mathscr{D}$, whence the result.

COROLLARY. *If $\mathscr{B}$ is a bivariety every sub-bivariety of which is generated by finite bigroups, then $\Lambda(\mathscr{B})$ is distributive if and only if $\Phi(\mathscr{B})$, $\Psi(\mathscr{B})$ are distributive.* (4.1.6)

The following notation will be used throughout this section.

NOTATION. *Let m, n be natural numbers greater than 1. Write $\mathscr{V} = \mathfrak{A}_m \circ \mathfrak{A}_n$, and to each prime p dividing m write $\mathscr{V}_p = \mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_n$, $\mathscr{U}_p = \mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\beta}$ where p^α, p^β are the maximum powers of p dividing m, n respectively; also put $n = p^\beta n_p$.* (4.1.7)

THEOREM. *Let $\mathscr{S}$ be a subvariety of $\mathscr{V}$ containing $\mathfrak{E} \circ \mathfrak{A}_n$. To each p dividing m and each t dividing n_p there exists a unique subvariety $\mathscr{S}_{pt}$ of $\mathscr{S}$ such that*

(i) $\mathscr{S}_{p1} \in \Lambda(\mathscr{U}_p)$, $\quad \mathscr{S}_{pt} \in \Phi(\mathscr{U}_p) \quad (p|m,\ 1 \neq t|n_p)$;

(ii) $t|u|n_p$ *implies* $\mathscr{S}_{pu} \leqslant \mathscr{S}_{pt}$;

(iii) $\mathscr{S} = \bigvee\{\mathscr{S}_{pt}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A} : p|m,\ t|n_p\}$. (4.1.8)

Before proving this result we need a lemma similar to (4.1.4), and, if the bigroups involved are thought of as groups, identical with a special case of result of Kovács & Newman (1966, (1.12)).

LEMMA. *Let $\{\boldsymbol{G}_i : i \in I\}$, $\{\boldsymbol{H}_j : j \in J\}$ be critical bigroups in $\mathfrak{A}_m \circ \mathfrak{A}_n$ $(m, n > 0)$, where each $\boldsymbol{G}_i$ is non-nilpotent, and each $\boldsymbol{H}_j$ is nilpotent. If $\boldsymbol{G}$ is critical and not nilpotent and*

$$\boldsymbol{G} \in \operatorname{svar}\{\boldsymbol{G}_i, \boldsymbol{H}_j : i \in I, j \in J\},$$

then
$$\boldsymbol{G} \in \operatorname{svar}\{\boldsymbol{G}_i : i \in I,\ |K|\,\big|\,|K_i|, \exp \sigma \boldsymbol{G} = \exp \sigma \boldsymbol{G}_i\} \qquad (4.1.9)$$

(in the notation of (3.2.1)).

Proof. Suppose first that q is a bilaw in all $\boldsymbol{G}_i$, $\boldsymbol{H}_j$ such that $p = \exp \sigma \boldsymbol{G}_i = \exp \sigma \boldsymbol{H}_j = \exp \sigma \boldsymbol{G}$. As usual we may suppose that either q is in $A_1(\boldsymbol{Q}_2)$ or in $A_2(\boldsymbol{Q}_2)$; write q' for q in the first case and for $[y_1, q]$ in the latter. If $m = p^\gamma m'$ (where $p \nmid m'$), then $q'^{m'}$ is a bilaw in all $\boldsymbol{G}_i$, $\boldsymbol{H}_j$ and therefore in $\boldsymbol{G}$. Since p does not divide m', q' is a bilaw in $\boldsymbol{G}$, and therefore q is a bilaw in $\boldsymbol{G}$, $A_1(\boldsymbol{G})$ being self-centralizing.

Without loss of generality, then, we may suppose that $\exp \sigma \boldsymbol{G}_i = \exp \sigma \boldsymbol{H}_j = p$ for all i, j. Then let q be a bilaw in all $\boldsymbol{G}_i$ such that $|K|\,\big|\,|K_i|$: again we may assume q is in $A_1(\boldsymbol{Q}_2)$ or in $A_2(\boldsymbol{Q}_2)$ and define q' as in the last paragraph. If $\{n_1, \ldots, n_u\} = \{|K_i| : i \in I,\ |K| \nmid |K_i|\}$ then

$$[q', z_r^{p^\beta n_1}, \ldots, z_{r+u}^{p^\beta n_u}]$$

is a bilaw in all $\boldsymbol{G}_i$, $\boldsymbol{H}_j$, where p^β is the largest power of p dividing n and r is chosen large enough to avoid z's which occur in q. However, since K acts fixed point free on $A_1(\boldsymbol{G})$, q' is a bilaw in $\boldsymbol{G}$ and, as before, q is a bilaw in $\boldsymbol{G}$.

Proof of (4.1.8). Suppose that $\mathscr{S}$ is a subvariety of $\mathscr{V}$, and make the following definition for each p dividing m and each t dividing n_p:

$$\mathscr{S}_{pt} = \operatorname{svar}\{\boldsymbol{F}^* : \boldsymbol{G} \in \mathscr{S} \text{ is critical, } |K| = t, \exp \sigma\boldsymbol{G} = p\},$$

where we interpret $\boldsymbol{F}^* = \boldsymbol{G}$, $K = 1$ in case G is a p-group. If $\boldsymbol{G}$ in $\mathscr{S}$ is critical with $|K| = u$, and t divides u, write $\hat{\boldsymbol{G}}$ for the sub-bigroup $(F\hat{K}, A, H\hat{K})$ of $\boldsymbol{G}$ where $\hat{K}$ is the subgroup of K of order t. By (3.4.7) there exist critical bigroups $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w$ with

$$|K_1| = \ldots = |K_w| = t, \quad \operatorname{svar}\{\boldsymbol{G}_1, \ldots, \boldsymbol{G}_w\} = \operatorname{svar} \hat{\boldsymbol{G}},$$

and
$$\operatorname{svar}\{\boldsymbol{F}_1^*, \ldots, \boldsymbol{F}_w^*\} = \operatorname{svar} \boldsymbol{F}^*.$$

It follows that $\mathscr{S}_{pu} \leqslant \mathscr{S}_{pt}$.

By virtue of (2.2.5), (iii) will be proved if we can show that for non-trivial t dividing n_p

$$\operatorname{svar}\{\boldsymbol{G} \in \mathscr{S} : \boldsymbol{G} \text{ critical, } |K| = t, \exp \sigma\boldsymbol{G} = p\} = \mathscr{S}_{pt}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A}. \tag{4.1.10}$$

That the left-hand side is contained in the right-hand side is obvious, and for the opposite inclusion suppose that $\hat{\boldsymbol{G}}$ in $\mathscr{S}_{pt}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A}$ is critical; if it is not nilpotent, then $\hat{\boldsymbol{F}}$ is in $\mathscr{S}_{pt}$, $|K|$ divides t, and (3.4.4) then ensures that $\hat{\boldsymbol{G}}$ belongs to the left-hand side of (4.1.10); and if $\hat{\boldsymbol{G}}$ is nilpotent we draw the same conclusion immediately, completing the proof of existence in (4.1.8).

Suppose that $\mathscr{S}'_{pt}(t|n_p)$ are subvarieties of $\mathscr{S}$ satisfying the conditions (i) to (iii) in (4.1.8). First, it is easy to see that

$$\mathscr{S}'_{p1} = \mathscr{S} \wedge \mathscr{U}_p = \mathscr{S}_{p1};$$

and secondly, suppose t is not 1 so that, by virtue of (3.4.7) and the definition of $\mathscr{S}_{pt}$, $\mathscr{S}_{pt}$ contains $\mathscr{S}'_{pt}$ for all p dividing m. Conversely suppose that $\boldsymbol{G}$ in $\mathscr{S}$ is critical with $|K| = t$ and $\exp \sigma\boldsymbol{G} = p$; then (3.4.7) and (4.1.9) show that

$$\boldsymbol{G} \in \bigvee\{\mathscr{S}'_{pu}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A} : t|u\},$$

from which it follows that $\boldsymbol{F}^*$ is in $\mathscr{S}'_{pt}$. Since $\mathscr{S}_{pt}$ is generated by all such $\boldsymbol{F}^*$ it must be contained in $\mathscr{S}'_{pt}$; this completes the proof of uniqueness in (4.1.8).

Corollary. *Let Δ_p be the set of positive divisors of n_p with the division ordering. Then if $|\Delta_p| = s_p$, $\Lambda(\mathscr{V}_p)$ is a subdirect product of Δ_p, $\Lambda(\mathscr{U}_p)$ and $s_p - 1$ copies of $\Phi(\mathscr{U}_p)$.* (4.1.11)

Proof. Let $\Lambda_p = \Lambda(\mathscr{U}_p) \times \Phi(\mathscr{U}_p)^{s_p-1}$ be the set of all functions f defined on Δ_p such that

$$f(1) \in \Lambda(\mathscr{U}_p), \quad f(t) \in \Phi(\mathscr{U}_p) \quad (1 \neq t|n_p)$$

(the order on Λ_p being component-wise), and let Λ'_p be the sublattice of Λ_p:

$$\Lambda'_p = \{f \in \Lambda_p : t|u|n_p \text{ implies } f(u) \leqslant f(t)\}.$$

Clearly Λ'_p is subdirect.

Define the mapping $\lambda_p : \Lambda(\mathscr{V}_p) \to \Delta_p \times \Lambda'_p$ by

$$\mathscr{S}\lambda_p = (t_0, f_0)$$

where
$$\mathscr{S} \wedge \mathfrak{E} \circ \mathfrak{A}_{n_p} = \mathfrak{E} \circ \mathfrak{A}_{t_0}$$

and
$$f_0(t) = \begin{cases} \mathscr{S}_{pt} & \text{if } t|t_0 \\ \mathfrak{E} \circ \mathfrak{E} & \text{if } t \nmid t_0. \end{cases}$$

That f_0 is in Λ_p' follows from (4.1.8) (i) to (ii); and from (iii) we conclude that λ_p is one-to-one and onto. Since λ_p is also inclusion preserving it follows that it is a lattice isomorphism.

Notice, that if for t in $\{0\} \cup \Delta_p$, we define λ_{pt} by

$$\mathscr{S}\lambda_{pt} = f_0(t), \quad \mathscr{S} \leqslant \mathscr{V}_p, \quad 0 \neq t, \quad \mathscr{S}\lambda_0 = t_0, \tag{4.1.12}$$

where $\mathscr{S}\lambda_p = (t_0, f_0)$, then λ_{pt} are homomorphisms and yield the sub-direct decomposition of $\Lambda(\mathscr{V}_p)$ described above.

THEOREM. *Define $\mu_p: \Lambda(\mathscr{V}) \to \Lambda(\mathscr{V}_p)$ by $\mathscr{B}\mu_p = \mathscr{B} \wedge \mathscr{V}_p$, for all p dividing m. Then μ_p are all homomorphisms and they provide a subdirect decomposition of $\Lambda(\mathscr{V})$.* (4.1.13)

Proof. That each μ_p is a meet-homomorphism is obvious. To show that it is a join-homomorphism we must show that, for subvarieties $\mathscr{B}$, $\mathscr{C}$ of $\mathscr{V}$

$$(\mathscr{B} \vee \mathscr{C})\,\mu_p \leqslant \mathscr{B}\mu_p \vee \mathscr{C}_p\mu_p$$

since the converse is clear. If $\mathbf{G}$ in $(\mathscr{B} \vee \mathscr{C}) \wedge \mathscr{V}_p$ is critical, and $A_1(\mathbf{G})$ is not 1, then (4.1.9) yields that $\mathbf{G}$ belongs to $(\mathscr{B} \wedge \mathscr{V}_p) \vee (\mathscr{C} \wedge \mathscr{V}_p)$ which is what we want; if $A_1(\mathbf{G}) = 1$ then

$$\mathbf{G} \in (\mathscr{B} \vee \mathscr{C}) \wedge \mathfrak{E} \circ \mathfrak{A}_n = (\mathscr{B} \vee \mathscr{C})\,\psi = \mathscr{B}\psi \vee \mathscr{C}\psi \leqslant \mathscr{B}\mu_p \vee \mathscr{C}\mu_p,$$

using (4.1.3). Finally note that for $\mathscr{B} \leqslant \mathscr{V}$,

$$\mathscr{B} = \bigvee \{\mathscr{B}\mu_p : p|m\}, \tag{4.1.14}$$

and therefore the theorem is proved.

THEOREM. *If $\mathscr{B}$ is a subvariety of $\mathfrak{A}_m \circ \mathfrak{A}_n$ then $\Lambda(\mathscr{B})$ is distributive if and only if $\Lambda(\mathscr{B})\,\mu_p\lambda_{pt}$ is distributive for each p dividing m and each t dividing n_p.* (4.1.15)

Proof. Use (4.1.13), (4.1.11) and (4.1.6).

THEOREM. *The join-irreducible subvarieties of $\mathfrak{A}_m \circ \mathfrak{A}_n$ are precisely those of $\mathfrak{E} \circ \mathfrak{A}_n$, those of $\mathscr{U}_p(p|m)$ and those of the type*

$$\mathscr{S}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A},$$

where $\mathscr{S}$, a non-trivial element of $\Phi(\mathscr{U}_p)$, is join-irreducible, and $t(>1)$ divides n_p. (4.1.16)

Proof. Use (4.1.14) and (4.1.8). It is clear at once that a join-irreducible $\mathscr{B}$ must have this general form; and it is easy to see from the embedding in (4.1.11) that $\mathscr{S}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A}$ is join-irreducible if and only if $\mathscr{S}$ is join-irreducible.

We shall prove in (4.2.29) that $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ has minimum condition. Standard lattice theory then yields that every subvariety of $\mathfrak{A}_m \circ \mathfrak{A}_n$ is a finite irredundant join of join-irreducibles, so (4.1.16) and (4.2.33) enable us to describe the non-nilpotent join-irreducibles in this way (in terms of the nilpotent ones), though with possible lack of uniqueness for such descriptions. In a modular lattice (as here) each element has a unique decomposition as an irredundant join of join-irreducibles if and only if the lattice is distributive. From this point of view (4.1.15) is a useful reduction theorem.

In one case at least we are able to determine precisely the nature of irredundant join decompositions, as the next theorem shows.

THEOREM. *The decomposition*

$$\mathscr{V} = \bigvee \{\mathscr{V}_p : p|m\}$$

is the one and only way that $\mathscr{V}$ can be written as an irredundant join of join-irreducibles. (4.1.17)

Proof. We show in (4.2.30) that the $\mathscr{U}_p$ are join-irreducible. Suppose that for some p dividing m

$$\mathscr{V}_p = \mathscr{S} \vee \mathscr{S}'.$$

From (4.1.8) we have that

$$\mathscr{U}_p = \mathscr{S}\lambda_{pn_p} \vee \mathscr{S}'\lambda_{pn_p},$$

so that $\mathscr{U}_p = \mathscr{S}\lambda_{pn_p}$, say, and so for all t dividing n_p, $\mathscr{S}\lambda_{pt} = \mathscr{U}_p$; and (4.1.8) then gives $\mathscr{S} = \mathscr{V}_p$.

Certainly then, $\mathscr{V}$ has a decomposition as an irredundant join of join-irreducibles. Suppose that

$$\mathscr{V} = \mathscr{B}_1 \vee \ldots \vee \mathscr{B}_s$$

is another such decomposition. Then, for p dividing m,

$$\mathscr{V}_p = \mathscr{B}_1\mu_p \vee \ldots \vee \mathscr{B}_s\mu_p,$$

whence, for some j in $\{1, \ldots, s\}$,

$$\mathscr{V}_p = \mathscr{B}_j\mu_p \leqslant \mathscr{B}_j.$$

That is, each $\mathscr{V}_p$ is contained in some $\mathscr{B}_j$; and each $\mathscr{B}_j$ does contain some $\mathscr{V}_p$ since otherwise it is redundant. Also since each $\mathscr{B}_j$ is join-irreducible we have from (4.1.14) that $\mathscr{B}_j = \mathscr{B}_j\mu_q$ for some q dividing m. Hence

$$\mathscr{V}_p \leqslant \mathscr{B}_j = \mathscr{B}_j\mu_q \leqslant \mathscr{V}_q.$$

It must follow that $p = q$ and $\mathscr{B}_j = \mathscr{V}_p$; this completes the proof.

Finally, the results proved so far in this section enable us to obtain a reduction theorem for minimum condition in $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$.

THEOREM. *$\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ has descending chain condition if and only if each $\Lambda(\mathscr{U}_p)$ $(p|m)$ has descending chain condition.* (4.1.18)

Proof. Since $\mathfrak{A}_m \circ \mathfrak{A}_n = \vee\{\mathscr{V}_p : p|m\}$, (2.1.3) shows that $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ has descending chain condition if and only if each $\Lambda(\mathscr{V}_p)$ does. Now let

$$\mathscr{V}_p = \mathscr{B}_0 \geqslant \mathscr{B}_1 \geqslant \ldots \geqslant \mathscr{B}_j \geqslant \ldots$$

be a descending chain in $\Lambda(\mathscr{V}_p)$. Then in the notation of (4.1.7) and (4.1.8), for each t dividing n_p we have that

$$(\mathscr{B}_0)_{pt} \geqslant (\mathscr{B}_1)_{pt} \geqslant \ldots \geqslant (\mathscr{B}_j)_{pt} \geqslant \ldots$$

is a descending chain. If each of these chains terminate, then for some natural number j_0,

$$j \geqslant j_0 \quad \text{implies} \quad (\mathscr{B}_j)_{pt} = (\mathscr{B}_{j_0})_{pt} \quad \text{for all} \quad t|n_p.$$

Hence (4.1.8) gives that

$$j \geqslant j_0 \quad \text{implies} \quad \mathscr{B}_j = \mathscr{B}_{j_0}.$$

As the converse is obvious, this completes the proof of (4.1.18).

4.2. $\Lambda(\mathfrak{A}_{p\alpha} \circ \mathfrak{A}_{p\beta})$: *preliminary lemmas*

Preparatory to our attack on $\Lambda(\mathfrak{A}_{p\alpha} \circ \mathfrak{A}_{p\beta})$ we introduce some necessary notation, and prove several technical lemmas. We shall for the most part be working with split-free bigroups and their normal fully invariant sub-bigroups rather than with bivarieties as such. As a matter of minor convenience we choose to work with $\mathfrak{A} \circ \mathfrak{A}_{p^\nu}$ (p prime), rather than with $\mathfrak{A}_{p\alpha} \circ \mathfrak{A}_{p^\nu}$. For lattice purposes it suffices to work in the split-free bigroup of $\mathfrak{A} \circ \mathfrak{A}_{p^\nu}$ of rank $(1, \infty)$, as the following result shows.

LEMMA. *The lattices of normal, fully invariant sub-bigroups of* $\boldsymbol{F}_{\infty}(\mathfrak{A}\circ\mathfrak{A}_{p^\nu})$ *and* $\boldsymbol{F}_{(1,\infty)}(\mathfrak{A}\circ\mathfrak{A}_{p^\nu})$ *are isomorphic.* (4.2.1)

Proof. We use (1.5.10), imagining $\boldsymbol{Q}(1,\infty)$ embedded in $\boldsymbol{Q}_2$ in a natural way. Consider the mapping ξ from the lattice of normal, fully invariant sub-bigroups of $\boldsymbol{Q}_2$ containing the bilaws $\mathfrak{A}\circ\mathfrak{A}_{p^\nu}(\boldsymbol{Q}_2)$ of $\mathfrak{A}\circ\mathfrak{A}_{p^\nu}$ to the lattice of normal, fully invariant sub-bigroups of $\boldsymbol{Q}(1,\infty)$ containing $\mathfrak{A}\circ\mathfrak{A}_{p^\nu}(\boldsymbol{Q}(1,\infty))$, defined by

$$\boldsymbol{S}\xi = S(\boldsymbol{Q}(1,\infty)).$$

Now ξ is onto by (1.4.13), clearly preserves inclusions, and by (1.5.10) is a meet-homomorphism; it is easy to see that ξ is then a join-homomorphism if it is one-to-one. If $\mathscr{S}_1$, $\mathscr{S}_2$ are distinct subvarieties of $\mathfrak{A}\circ\mathfrak{A}_{p^\nu}$ then there exists q in $(\boldsymbol{Q}(1,\infty)\cap\boldsymbol{S}_1)-\boldsymbol{S}_2$, say, by virtue of (2.2.3) and so, from (1.5.10), $S_1(\boldsymbol{Q}(1,\infty))$ and $S_2(\boldsymbol{Q}(1,\infty))$ are different. This completes the proof.

NOTATION. *Write* $\boldsymbol{W}_\nu$ *for* $\boldsymbol{F}_{(1,\infty)}(\mathfrak{A}\circ\mathfrak{A}_{p^\nu})$, $A=A_1(\boldsymbol{W}_\nu)$, $B=A_2(\boldsymbol{W}_\nu)$. *For the split-free generating set of* $\boldsymbol{W}_\nu$ *write* $\{y_1\}\cup\{z_1,z_2,\ldots,z_i,\ldots\}$; *no confusion will result from this. We will abuse language to the extent of calling elements of* $\boldsymbol{W}_\nu$ *biwords.* (4.2.2)

From theorem 1.7.12 we have that

$$W_\nu = C\,\mathrm{wr}\,F_\infty(\mathfrak{A}_{p^\nu}),$$

where C is an infinite cycle, and where A is the base group of W_ν, and $B=F_\infty(\mathfrak{A}_{p^\nu})$. We shall prove, *inter alia*,

THEOREM. *All ascending chains of normal, fully invariant sub-bigroups of* $\boldsymbol{W}_\nu$ *break off.* (4.2.3)

The results obtained in proving this will enable us to get at the join-irreducible sub-bivarieties of $\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^\beta}$.

It is worth noting here

LEMMA. *Every fully invariant sub-bigroup of* $\boldsymbol{W}_\nu$ *contained in* A *is normal in* $\boldsymbol{W}_\nu$. (4.2.4)

Proof. This follows since elements of B induce self-morphisms of $\boldsymbol{W}_\nu$, and A is abelian.

LEMMA. *If* U *is a normal sub-bigroup of* $\boldsymbol{W}_\nu$, *and if for fixed elements* $a_1,\ldots,a_m$ *in* A, *and all* b *in* B

$$\prod_{i=1}^{m}[a_i,b^i]\in U$$

then for all $b_1,\ldots,b_m$ *in* B, $\Pi_{j=u}^m[a_j,b_1^j,\ldots,b_u^{j-u+1}]$ *belongs to* U, $u\in\{1,\ldots,m\}$. (4.2.5)

Proof. For $u=1$ the assertion is the hypothesis. Suppose, therefore, that for some $u\in\{1,\ldots,m-1\}$ the lemma is true. If $b_1,\ldots,b_{u+1}$ are arbitrarily chosen elements of B, then

$$\prod_{j=u}^{m}[a_j,b_1^j,\ldots,(b_u b_{u+1})^{j-u+1}]\in U.$$

That is,
$$\prod_{j=u}^{m}[a_j,b_1^j,\ldots,b_u^{j-u+1}]\,[a_j,b_1^j,\ldots,b_{u-1}^{j-u+2},b_{u+1}^{j-u+1}]\,[a_j,b_1^j,\ldots,b_u^{j-u+1},b_{u+1}^{j-u+1}]\in U$$

and from here, using our inductive hypothesis, we obtain that

$$\prod_{j=u}^{m}[a_j,b_1^j,\ldots,b_u^{j-u+1},b_{u+1}^{j-u+1}]\in U. \tag{4.2.6}$$

Since U is normal we have $\Pi_{j=u}^m[a_j,b_1^j,\ldots,b_u^{j-u+1},b_{u+1}]\in U$ and so

$$\prod_{j=u}^{m}[a_j,b_1^j,\ldots,b_u^{j-u+1},b_{u+1}]^{-1}[a_j,b_1^j,\ldots,b_u^{j-u+1},b_{u+1}^{j-u+1}]\in U.$$

Finally, using the commutator identity $[x, y]^{-1}[x, y^t] = [x, y^{t-1}]^y$ for all integers t, we have

$$\prod_{j=u+1}^{m} [a_j, b_1^j, \ldots, b_{u+1}^{j-u}]^{b_{u+1}} \in U,$$

which, since U is normal, gives what we want.

This lemma will prove useful in a number of places: first as the initial step of an induction in the proof of lemma 4.2.10 below, and later in dealing with the structure of certain metabelian varieties.

Notation. *If* $\mathbf{U}$ *is normal in* $\mathbf{W}_v$ *define the sub-bigroups* $\mathbf{U}_i$ *of* $\mathbf{W}_v$ *for non-negative integers i by*

$$U_i/U = Z_i(W_v/U),$$

where $Z_i(W_v/U)$ *is the i-th term of the upper central series of* W_v/U *(see, for example, p.* 77 *in* Hanna Neumann 1967).

Note that if a is in A, then $[a, b_1, \ldots, b_r]$ belongs to U for all $b_1, \ldots, b_r$ in B if and only if a is in U_r.

Lemma. *If to the hypotheses of* (4.2.5) *we add* $m \leqslant p-1$, *then for* $i \in \{1, \ldots, m\}$, $a_i \in U_m$. (4.2.7)

Proof. From (4.2.5),

$$[a_m, b_1^m, \ldots, b_{m-1}^2, b_m] \in U$$

for all $b_1, \ldots, b_m$ in B. Since 1, 2, ..., m are all prime to p, we have $a_m \in U_m$.

Assume that it has been proved that $a_{i+1} \in U_m, \ldots, a_m \in U_m$ for some $i \geqslant 1$. Then since $\Pi_{j=i}^{m}[a_j, b_1^j, \ldots, b_i^{j-i+1}] \in U$, we have by commuting with $b_{i+1}, \ldots, b_m$ that $[a_i, b_1^i, \ldots, b_i, b_{i+1}, \ldots, b_m] \in U$ and hence, as before, $a_i \in U_m$. This completes the proof.

Lemma. *If* $\mathbf{U}$ *is normal in* $\mathbf{W}_v$ *and if for fixed elements* $a_1, \ldots, a_m$ *of* A *and all* b *in* B,

$$\rho = \prod_{i=1}^{m} [a_i, b^i] \in U$$

then for all b *in* B,

(i) $([a_t, b^p][a_{t+p}, b^{2p}] \ldots) \in U_{m-1}, \quad p \leqslant t \leqslant 2p-1;$

(ii) $([a_u, b^u][a_{u+p}, b^{u+p}] \ldots) \in U_{m+p-2}, \quad 1 \leqslant u \leqslant p-1;$

(iii) $(a_v a_{v+p} \ldots) \in U_{m+p-2}, \quad 1 \leqslant v \leqslant p-1.$ (4.2.8)

In the proof of this lemma we need the following notation, and lemma 4.2.10 below.

Notation. *If* $b_1, \ldots, b_m$ *are arbitrary elements of* B, *write*

$$c(s, u, v, i) = [a_{s+ip}, b_1^{s+ip}, \ldots, b_{s-up+v}^{up-v+ip+1}, b_{s-(u-1)p+1}^{(u+i-1)p}, \ldots, b_{s-p+1}^{(i+1)p}]$$

where $s \in \{1, \ldots, m\}$, $i \in \{0, \ldots, l_s\}$ *where* $l_s = [(m-s)/p]$, $v \in \{1, \ldots, p\}$ *and where u has the range*:

$$u \in \{1, \ldots, s/p\} \quad \text{if} \quad p|s,$$
$$u \in \{1, \ldots, [s/p]+1\} \quad \text{if} \quad p \nmid s,$$

with the conventions:

$s - up + v \leqslant 0$ *implies*

$$c(s, u, v, i) = [a_{s+ip}, b_{s-(u-1)p+1}^{(u+i-1)p}, \ldots, b_{s-p+1}^{(i+1)p}];$$

$s - up + v \leqslant s < s-(u-1)p+1$ *implies*

$$c(s, u, v, i) = [a_{s+ip}, b_1^{s+ip}, \ldots, b_{s-up+v}^{up-v+ip+1}].$$

Also write

$$\rho(s, u, v) = \prod_{i=0}^{l_s} c(s, u, v, i). \tag{4.2.9}$$

LEMMA. *If* ρ *is as in* (4.2.8) *then*

$$\rho(s, u, v) \in U_r$$

for all relevant s, u, v, *where* $r = m-s+u(p-1)-v+1.$ (4.2.10)

Proof. From (4.2.5) we have

$$\rho(m, 1, p) = [a_m, b_1^m, \ldots, b_{m-1}^2, b_m] \in U;$$

and in this expression we may replace b_i^{m-i+1} by b_i whenever p does not divide $m-i+1$. Hence, since

$$\rho(m, u, v) = [a_m, b_1^m, \ldots, b_{m-up+v}^{up-v+1}, b_{m-(u-1)p+1}^{(u-1)p}, \ldots, b_{m-p+1}^{p}]$$

for all relevant u, v,

$$\rho(m, u, v) \in U_r,$$

where $r = m-(m-up+v+u-1) = u(p-1)-v+1$. We use this as the start of an induction, the induction being taken over the lexicographically ordered set of triples $(-s, u, -v)$. Suppose, therefore, that for all $(-s, u, -v) < (-t, w, -x+1)$ where $x \in \{2, \ldots, p\}$, the assertion of the lemma is true.

First note that from lemma 4.2.5 we have

$$\prod_{j=t}^{m} [a_j, b_1^j, \ldots, b_t^{j-t+1}] = \prod_{j=t}^{t+p-1} \rho(j, 1, t+p-j) \in U.$$

Hence, by the inductive hypothesis we deduce from this that

$$\rho(t, 1, p) \in U_{m-t}$$

as required. Secondly,

$$\rho(t, w, x) = \prod_{i=0}^{l_t} c(t, w, x, i)$$

and
$$\begin{aligned} c(t, w, x, i) &= [a_{t+ip}, b_1^{t+ip}, \ldots, b_{t-wp+x}^{wp-x+ip+1}, b_{t-(w-1)p+1}^{(w+i-1)p}, \ldots, b_{t-p+1}^{(i+1)p}] \\ &= [a_{t+ip}, b_1^{t+ip}, \ldots, b_{t-wp+x-1}^{wp-x+ip+2}, b_{t-(w-1)p+1}^{(w+i-1)p}, \ldots, b_{t-p+1}^{(i+1)p}, b_{t-wp+x}^{wp-x+ip+1}] \\ &= [c(t, w, x-1, i), b_{t-wp+x}^{wp-x+1}] \\ &\quad \times [a_{t+ip}, b_1^{t+ip}, \ldots, b_{t-wp+x-1}^{wp-x+ip+2}, b_{t-(w-1)p+1}^{(w+i-1)p}, \ldots, b_{t-p+1}^{(i+1)p}, b_{t-wp+x}^{ip}]^{b_{t-wp+x}^{wp-x+1}}. \end{aligned}$$

Therefore

$$\rho(t, w, x) = [\rho(t, w, x-1), b_{t-wp+x}^{wp-x+1}] \cdot \rho'(t+p, w+1, x-1)^{b_{t-wp+x}^{wp-x+1}},$$

where $\rho'(t+p, w+1, x-1)$ differs from $\rho(t+p, w+1, x-1)$ only in that the element $b_{(t+p)-p+1}$ occurs as b_{t-wp+x}; in any event $\rho'(t+p, w+1, x-1)^{b_{t-wp+x}^{wp-x+1}}$ belongs to U_r where

$$r = m-(t+p)+(w+1)(p-1)-(x-1)+1 = m-t+w(p-1)-x+1,$$

by the induction hypothesis. Hence, since also $\rho(t, w, x)$ is in U_r by the inductive hypothesis,

$$[\rho(t, w, x-1), b_{t-wp+x}^{wp-x+1}] \in U_r;$$

and the fact that $wp-x+1$ is prime to p under the assumption on x, and that b_{t-wp+x} does not occur in $\rho(t, w, x-1)$, means that

$$\rho(t, w, x-1) \in U_{r+1},$$

as required.

Finally, note that for $u \geqslant 2$,

$$\rho(s, u, p) = \rho(s, u-1, 1)$$

and this completes the induction, and the proof of (4.2.10).

Proof of (4.2.8). Put $s = p$, $u = 1$, $v = 1$ in (4.2.10) and we get

$$\prod_{i=0}^{l_p} [a_{(i+1)p}, b_1^{(i+1)p}] \in U_{m-1}.$$

If $p < s \leqslant 2p-1$, put $u = 2$, $v = 2p-s$ and we get

$$\prod_{i=0}^{l_s} [a_{s+ip}, b_{s-p+1}^{(i+1)p}] \in U_{m-1},$$

and these together are just the assertion (i).

To prove (iii) proceed as follows. We have

$$\rho = \prod_{i=1}^{p-1} [a_i, b^i] \prod_{j=p}^{2p-1} \prod_{k=0}^{l_j} [a_{j+kp}, b^{j+kp}]$$
$$= \prod_{i=1}^{p-1} [a_i, b^i] \prod_{j=p}^{2p-1} \left\{ \prod_{k=0}^{l_j} [a_{j+kp}, b^{(k+1)p}]^{b^{j-p}} \left[\prod_{k=0}^{l_j} a_{j+kp}, b^{j-p} \right] \right\}.$$

By part (i), then,

$$\prod_{i=1}^{p-1} [a_i, b^i] \prod_{j=p+1}^{2p-1} \left[\prod_{k=0}^{l_j} a_{j+kp}, b^{j-p} \right] \in U_{m-1},$$

and therefore

$$\prod_{i=1}^{p-1} \left[\prod_{k=0}^{l_i} a_{i+kp}, b^i \right] \in U_{m-1}.$$

Lemma 4.2.7 yields, for all $i \in \{1, \dots, p-1\}$,

$$\prod_{k=0}^{l_i} a_{i+kp} \in U_{m+p-2}$$

and this completes the proof of (iii).

The proof of (ii) uses (i), (iii) and the identity

$$\prod_{k=0}^{l_v} [a_{v+kp}, b^{v+kp}] = \left[\prod_{k=0}^{l_v} a_{v+kp}, b^v \right] \prod_{k=1}^{l_v} [a_{v+kp}, b^{kp}]^{b^v} \tag{4.2.11}$$

for $v \in \{1, \dots, p-1\}$. The proof of (4.2.8) is now complete.

Definition. *An element of W_ν which belongs to the subgroup generated by the set*

$$\{y_1\} \cup \{z_1^p, z_2^p, \dots, z_i^p, \dots\}$$

will be called a †-biword. (4.2.12)

Lemma. *If $q \in A$, then there exist †-biwords $q_1, \dots, q_d$ and a natural number v such that*

(i) $q \in \langle\{q_1, \dots, q_d\}\rangle^{W_\nu}$,

(ii) $[q_i, vB] \leqslant cl\{q\}$ $(1 \leqslant i \leqslant d)$.

Moreover, if q is uniform, so are $q_1, \dots, q_d$. (As usual, $[q_i, vB]$ stands for the subgroup generated by the commutators $[q_i, b_1, \dots, b_v]$, $b_1, \dots, b_v \in B$.) (4.2.13)

The proof of this lemma depends on the following consideration.

Lemma. *If $q^* \in A$ is a uniform biword, say involving the variables $y_1, z_1, \dots, z_s$ precisely, then there exist uniform biwords $q_1^*, \dots, q_r^*$ in which z_s, if it occurs at all, does so raised to a power which is a multiple of p, and $q_1^*, \dots, q_r^*$ involve no variables other than $y_1, z_1, \dots, z_s$; and there exists a natural number v^* such that*

$$\left.\begin{array}{l} \text{(i)}\ \ q^* \in \langle\{q_1^*, \dots, q_r^*\}\rangle^{W_\nu}, \\ \text{(ii)}\ \ [q_i^*, v^* B] \leqslant cl\{q^*\} \quad (1 \leqslant i \leqslant r). \end{array}\right\} \tag{4.2.14}$$

Proof. We may write

$$q^* = \prod_{i=1}^{t} [y_1, z_1^{\lambda_{i1}}, \ldots, z_s^{\lambda_{is}}]^{\alpha_i},$$

where $0 < \lambda_{ij} \leqslant p^\nu - 1$ for all i, j. For $j \in \{1, \ldots, p^\nu - 1\}$ define

$$a_j = \begin{cases} \prod_{j=\lambda_{is}} [y_1, z_1^{\lambda_{i1}}, \ldots, z_{s-1}^{\lambda_{is-1}}]^{\alpha_i}, & \exists\, i, j = \lambda_{is}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $q^* = \prod_{i=1}^{p^\nu - 1} [a_j, z_s^j]$. Since by construction the a_j's do not involve z_s, the hypotheses of lemma 4.2.8 are satisfied, with $U = cl\{q^*\}$. Hence, if $u' = [(p^\nu - 1 - u)/p]$ and $v' = [(p^\nu - 1 - v)/p]$,

$$q_u^{**} = \prod_{k=0}^{u'} [a_{u+kp}, z_s^{(k+1)p}] \in U_{p^\nu - 2} \quad (p \leqslant u \leqslant 2p-1),$$

$$q_v^{**} = \prod_{k=0}^{v'} a_{v+kp} \in U_{p^\nu + p - 3} \quad (1 \leqslant v \leqslant p-1).$$

By virtue of the fact that

$$q^* = \prod_{v=1}^{p} \prod_{k=0}^{v'} [a_{v+kp}, z_s^{v+kp}],$$

and from (4.2.11), we have

$$q^* \in \langle \{q_i^{**} : 1 \leqslant i \leqslant 2p-1\} \rangle^{W_\nu}.$$

Put $\{q_1^*, \ldots, q_r^*\} = \{q_i^{**} : q_i^{**} \neq 1, 1 \leqslant i \leqslant 2p-1\}$, $v^* = p^\nu + p - 3$, and we are finished.

Proof of (4.2.13). We can, by virtue of (1.4.5), assume q to be uniform. Then apply (4.2.14) to q, say q involves precisely $y_1, z_1, \ldots, z_s$, and obtain $q_1^*, \ldots, q_r^*$ in which z_s occurs either not at all, or to a power which is a multiple of p. Then use (4.2.14) on $q_1^*, \ldots, q_r^*$, first moving z_{s-1} up to the back of each commutator, and making z_{s-1} 'good' according to (4.2.14). Continue this process until we have dealt with $z_s, \ldots, z_1$ in turn, and hence reached a set of †-biwords $q_1, \ldots, q_d$ and a natural number v (the sum of all the relevant v^*'s) which satisfy the assertions of the lemma.

LEMMA. *Suppose that* $\boldsymbol{U}$ *is a biverbal sub-bigroup of* $\boldsymbol{W}_\nu$ *determined by †-biwords. Then* $\boldsymbol{U} = \boldsymbol{U}_i$ *for all i in* I^+. (4.2.15)

Proof. We may suppose that the †-biwords determining $\boldsymbol{U}$ are

$$q_i = \prod_{j=1}^{t} [y_1, z_1^{p\lambda_{ij1}}, \ldots, z_{s_i}^{p\lambda_{ijs_i}}]^{\alpha_{ij}}, \quad i \in I,$$

where $\lambda_{ijk} > 0$. Clearly it suffices to show that if $q' = [q, z_d]$ is in $\boldsymbol{U}$ (where q involves $y_1, z_1, \ldots, z_{d-1}$ at most) then q is in $\boldsymbol{U}$.

There are values of the biwords $q_i, v_1, \ldots, v_N$ say, such that

$$q' = v_1 v_2 \ldots v_N. \tag{4.2.16}$$

Each v_j is obtained from some q_i by substituting for y_1 an element of A, and for $z_1, \ldots, z_{s_i}$, elements of B. By applying to (4.2.16) the deletion δ_{2d} if necessary, we may suppose that each v_j involves z_d. These z_d's entered v_j by substitution in some q_i either for y_1 or for some z_k; in the latter case the relevant z_d's will occur raised to a power which is a multiple of p. Consider the self-morphism μ of $\boldsymbol{W}_\nu$ defined by

$$y_1\mu = y_1, \quad z_j\mu = z_j, \quad j \neq d, \quad z_d\mu = z_d^{p^\nu - 1}.$$

Then, under μ, (4.2.16) becomes

$$(q'\mu)\,(v_{j_1}^{-1}\mu) \ldots (v_{j_w}^{-1}\mu) = 1, \tag{4.2.17}$$

where v_{j_k} involves z_d only by virtue of the substitution for y_1 in the relevant q_i. Indeed, since the commutators involved in the expressions for q_i are linear in the first entry, we may suppose, by renaming if necessary, that v_{j_k} is obtained from some q_i by a substitution for y_1 of a power of a single commutator of the form

$$[y_1, z_{d_1}^{\delta_1}, \ldots, z_{d_u}^{\delta_u}, z_d^{\delta}],$$

where $d_1, \ldots, d_u, d$ are distinct, and where δ is not divisible by p, and some unspecified substitution for $z_1, \ldots, z_{si}$ (though it does not involve z_d). That is, there exist values $v'_1, \ldots, v'_R$ of the q_i which do not involve z_d at all, such that

$$(q'\mu)\,[v'_1, z_d^{\zeta_1 p^{\nu-1}}] \ldots [v'_R, z_d^{\zeta_R p^{\nu-1}}] = 1, \tag{4.2.18}$$

with $1 \leqslant \zeta_1 \leqslant \ldots \leqslant \zeta_R \leqslant p-1$, say.

Lemma 4.2.7, or at any rate the same proof exactly, can now be used to conclude that

$$[q \prod_{\zeta_i=1} v'_i, z_d^{p^{\nu-1}}, \ldots, z_{d+p-2}^{p^{\nu-1}}] = 1. \tag{4.2.19}$$

By a result of Baumslag (1959) (24.22 in Hanna Neumann 1967),

$$q \prod_{\zeta_i=1} v'_i = 1,$$

and in consequence, q belongs to U.

Write Λ_ν for the lattice of normal, fully invariant sub-bigroups of $\boldsymbol{W}_\nu$. We aim to show that the †-biwords provide an embedding of $\Lambda_{\nu-1}$ into Λ_ν.

Suppose, therefore, that $\boldsymbol{W}_{\nu-1}$ is free on $\{\hat{y}_1\} \cup \{\hat{z}_1, \hat{z}_2, \ldots\}$, that $\hat{A} = A_1(\boldsymbol{W}_{\nu-1})$, $\hat{B} = A_2(\boldsymbol{W}_{\nu-1})$ and that the morphism $\xi_\nu : \boldsymbol{W}_{\nu-1} \to \boldsymbol{W}_\nu$ is defined by

$$\hat{y}_1 \xi_\nu = y_1, \quad \hat{z}_j \xi_\nu = z_j^p, \quad j \in I^+.$$

The morphism ξ_ν induces a mapping $\lambda_\nu : \Lambda_{\nu-1} \to \Lambda_\nu$ in the following natural way: if $\boldsymbol{L} \in \Lambda_{\nu-1}$, that is, if $\boldsymbol{L}$ is normal and fully invariant in $\boldsymbol{W}_{\nu-1}$, then

$$\boldsymbol{L}\lambda_\nu = cl\{l\xi_\nu : l \in \boldsymbol{L}\}. \tag{4.2.20}$$

It is clear at once that λ_ν is a join-homomorphism, but not so clear that it is a meet-homomorphism. In fact we prove

Lemma. *The mapping $\lambda_\nu : \Lambda_{\nu-1} \to \Lambda_\nu$ is a one-to-one lattice homomorphism.* (4.2.21)

Proof. First note that λ_ν preserves inclusion. We are left to show that λ_ν is a meet-homomorphism and that it is one-to-one. To prove the former it suffices to prove that for $\boldsymbol{L}_1, \boldsymbol{L}_2 \in \Lambda_{\nu-1}$,

$$\boldsymbol{L}_1\lambda_\nu \cap \boldsymbol{L}_2\lambda_\nu \leqslant (\boldsymbol{L}_1 \cap \boldsymbol{L}_2)\,\lambda_\nu, \tag{4.2.22}$$

since the opposite inclusion is obvious. We need several lemmas to do this.

Lemma. *If $\boldsymbol{L} \in \Lambda_{\nu-1}$, then $\boldsymbol{L}\lambda_\nu = (\boldsymbol{L}\xi_\nu)^{W_\nu}$.* (4.2.23)

Proof. Let $\alpha : \boldsymbol{W}_\nu \to \boldsymbol{W}_\nu$, then if $\beta : B \to \hat{B}$ is defined by

$$z_j \beta = \hat{z}_j, \quad j \in I^+,$$

define $\hat{\alpha} : \boldsymbol{W}_{\nu-1} \to \boldsymbol{W}_{\nu-1}$ by

$$\hat{y}_1 \hat{\alpha} = \hat{y}_1, \quad \hat{z}_j \hat{\alpha} = (z_j \alpha)\,\beta.$$

Also define $\alpha_1 : \boldsymbol{W}_\nu \to \boldsymbol{W}_\nu$ by

$$y_1 \alpha_1 = y_1 \alpha, \quad z_j \alpha_1 = z_j, \quad j \in I^+.$$

Then if $l \in \boldsymbol{L}$, $(l\xi_\nu)\,\alpha = (l\hat{\alpha})\,\xi_\nu \alpha_1 \in (\boldsymbol{L}\xi_\nu)\,\alpha_1 \leqslant (\boldsymbol{L}\xi_\nu)^{W_\nu}$ as required. This last inclusion is seen from the fact that every normal subgroup of W_ν admits α_1.

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LEMMA. *If* $\mathbf{L} \in \Lambda_{\nu-1}$, *then* (*writing* $L\lambda_\nu$ *for the carrier of* $\mathbf{L}\lambda_\nu$)

$$L\lambda_\nu \cap A = (L \cap \hat{A})\,\lambda_\nu, \quad L\lambda_\nu \cap B = (L \cap \hat{B})\,\xi_\nu. \tag{4.2.24}$$

Proof. Put $L \cap \hat{A} = A_0$, $L \cap \hat{B} = B_0$. Then

$$B_0\lambda_\nu = [A, B_0\,\xi_\nu]\,(B_0\,\xi_\nu)$$

and $[\hat{A}, B_0]\,\lambda_\nu$ carries $cl\{[\hat{a}, b_0]\,\xi_\nu : \hat{a} \in \hat{A}, b_0 \in B_0\}$ which is equal to $cl\{[y_1, b_0\,\xi_\nu] : b_0 \in B_0\}$ and this in turn is carried by $[A, B_0\,\xi_\nu]$, a subgroup of $A_0\lambda_\nu$. Hence

$$L\lambda_\nu = (A_0\lambda_\nu)\,(B_0\lambda_\nu) = (A_0\lambda_\nu)\,(B_0\,\xi_\nu),$$

so that
$$L\lambda_\nu \cap A = A_0\lambda_\nu = (L \cap \hat{A})\,\lambda_\nu, \quad L\lambda_\nu \cap B = B_0\,\xi_\nu = (L \cap \hat{B})\,\xi_\nu.$$

From this lemma, and from the definition of ξ_ν, we have that

$$\begin{aligned}(L_1\lambda_\nu \cap L_2\lambda_\nu) \cap B &= (L_1\lambda_\nu \cap B) \cap (L_2\lambda_\nu \cap B)\\ &= (L_1 \cap \hat{B})\,\xi_\nu \cap (L_2 \cap \hat{B})\,\xi_\nu = (L_1 \cap L_2 \cap \hat{B})\,\xi_\nu\\ &= (L_1 \cap L_2)\,\lambda_\nu \cap B.\end{aligned}$$

Hence in order to prove (4.2.22) it suffices to show that $L_1\lambda_\nu \cap L_2\lambda_\nu \cap A$ is contained in $(L_1 \cap L_2)\,\lambda_\nu \cap A$, or that

$$(L_1 \cap \hat{A})\,\lambda_\nu \cap (L_2 \cap \hat{A})\,\lambda_\nu \leqslant (L_1 \cap L_2 \cap \hat{A})\,\lambda_\nu. \tag{4.2.25}$$

If q belongs to the left-hand side of (4.2.25) then, by virtue of (4.2.13), there exist †-biwords $q_1, \ldots, q_d$, and an integer v such that

$$[q_i, vB] \leqslant (L_1 \cap \hat{A})\,\lambda_\nu \cap (L_2 \cap \hat{A})\,\lambda_\nu, \quad i \in \{1, \ldots, d\}.$$

However the sub-bigroups carried by $(L_1 \cap \hat{A})\,\lambda_\nu$, $(L_2 \cap \hat{A})\,\lambda_\nu$ are determined by †-biwords and therefore lemma 4.2.15 ensures that for each i, q_i belongs to $(L_1 \cap \hat{A})\,\lambda_\nu \cap (L_2 \cap \hat{A})\,\lambda_\nu$. The other piece of information from (4.2.13) is that q is in $cl\{q_1, \ldots, q_d\}$ hence the sub-bigroup carried by $(L_1 \cap \hat{A})\,\lambda_\nu \cap (L_2 \cap \hat{A})\,\lambda_\nu$ is determined by †-biwords.

In order to finish off the proof of (4.2.22) we need the following lemma. The proof given is due to L. G. Kovács, and replaces my original, much longer, proof.

LEMMA. *If* $\mathbf{L} \in \Lambda_{\nu-1}$, $L \leqslant \hat{A}$, *and if* $q \in \mathbf{L}\lambda_\nu$ *is a* †*-biword, then* $q \in \mathbf{L}\xi_\nu$. (4.2.26)

Proof. By (4.2.23), $q \in (\mathbf{L}\xi_\nu)^{W_\nu}$ and hence there exist l_i in L and b_i in B such that

$$q = \prod_{i=1}^{t} (l_i\,\xi_\nu)^{b_i}.$$

Write T for a fixed transversal of B^p in B (with $1 \in T$). Then $b_i = b'_i b''_i$ ($b'_i \in B^p$, $b''_i \in T$), and

$$\begin{aligned}q &= \prod_{b \in T} \Big(\prod_{b''_i = b} (l_i\,\xi_\nu)^{b'_i} \Big)^b\\ &= \prod_{b \in T} \Big(\prod_{b''_i = b} (l_i^{b'_i\xi_\nu^{-1}})\,\xi_\nu \Big)^b\\ &= \prod_{b \in T} (l_b\,\xi_\nu)^b \quad \text{where} \quad l_b \in L.\end{aligned}$$

Note that q, $l_b\,\xi_\nu$ all belong to $W_{\nu-1}\,\xi_\nu \cap A$ and therefore each has its support contained in B^p. However, supp $(l_b\,\xi_\nu)^b$ is contained in $B^p b^{-1}$, and since these cosets are pairwise disjoint,

$$\operatorname{supp} q = \bigcup_{b \in T} \operatorname{supp}\,(l_b\,\xi_\nu)^b \subseteq B^b,$$

whence, if b is a non-trivial element of T, supp $l_b\,\xi_\nu$ is empty, and so $l_b = 1$; thus $q = l_1\,\xi_\nu$ and therefore belongs to $\mathbf{L}\xi_\nu$.

To complete the proof of (4.2.22) observe that the sub-bigroup carried by

$$(L_1 \cap \hat{A})\,\lambda_\nu \cap (L_2 \cap \hat{A})\,\lambda_\nu$$

is determined by †-biwords, one of which is q', say. Lemma 4.2.26 shows that

$$q' \in (L_1 \cap \hat{A})\,\xi_\nu \cap (L_2 \cap \hat{A})\,\xi_\nu = (L_1 \cap L_2 \cap \hat{A})\,\xi_\nu \leqslant (L_1 \cap L_2 \cap \hat{A})\,\lambda_\nu.$$

This completes the proof of (4.2.22).

To finish off the proof of (4.2.21) we need to show that λ_ν is one-to-one. If $\boldsymbol{L}_1\lambda_\nu = \boldsymbol{L}_2\lambda_\nu$ then $L_1\lambda_\nu \cap B = L_2\lambda_\nu \cap B$ so that, from (4.2.24), $(L_1 \cap \hat{B})\,\xi_\nu = (L_2 \cap \hat{B})\,\xi_\nu$ whence $L_1 \cap \hat{B} = L_2 \cap \hat{B}$. Also $L_1\lambda_\nu \cap A = L_2\lambda_\nu \cap A$ and therefore, by (4.2.24), $(L_1 \cap \hat{A})\,\lambda_\nu = (L_2 \cap \hat{A})\,\lambda_\nu$. Now $(L_2 \cap \hat{A})\,\lambda_\nu$ is determined by †-biwords $l\xi_\nu (l \in L_2 \cap \hat{A})$, and (4.2.26) then gives that $l\xi_\nu$ belongs to $(L_1 \cap \hat{A})\,\xi_\nu$, or that l belongs to $L_1 \cap \hat{A}$. That is, $L_2 \cap \hat{A}$ is contained in $L_1 \cap \hat{A}$. In a similar way we prove that $L_2 \cap \hat{A}$ contains $L_1 \cap \hat{A}$ and therefore $L_1 \cap \hat{A} = L_2 \cap \hat{A}$. Hence $\boldsymbol{L}_1 = \boldsymbol{L}_2$, and this completes the proof of (4.2.21).

We now derive some properties of the embedding λ_ν which are essentially extensions of (4.2.13).

Lemma. *If $\Lambda_{\nu-1}$ has ascending chain condition then to each $\boldsymbol{U}$ in Λ_ν with U contained in A, there corresponds a unique $\boldsymbol{L}$ in $\Lambda_{\nu-1}$, with L contained in $\hat{A}$, and a natural number $v = v(\boldsymbol{U})$ such that*

$$[L\lambda_\nu, vB] \leqslant U \leqslant L\lambda_\nu. \tag{4.2.27}$$

Proof. To each q in U associate the †-biwords $q_1, \ldots, q_d$ of (4.2.13) and the natural numbers, v_q say, involved there. If $\boldsymbol{S}_q = cl\{q_1, \ldots, q_d\}$ then

$$[S_q, v_q B] \leqslant cl\{q\} \leqslant S_q.$$

As the q_i are †-biwords, there exists $\boldsymbol{L}_q$ in $\Lambda_{\nu-1}$ with $\boldsymbol{L}_q\lambda_\nu = \boldsymbol{S}_q$. Write

$$\boldsymbol{L} = \Pi\{\boldsymbol{L}_q : q \in \boldsymbol{U}\}.$$

Since $\Lambda_{\nu-1}$ is assumed to have ascending chain condition, $\boldsymbol{L}$ is the join of finitely many $\boldsymbol{L}_q$'s, say, of those corresponding to q in the finite set X. Put $v = \max\{v_q : q \in X\}$ and then

$$\boldsymbol{U} \leqslant \Pi\{\boldsymbol{S}_q : q \in \boldsymbol{U}\} = \Pi\{\boldsymbol{L}_q\lambda_\nu : q \in \boldsymbol{U}\} = \boldsymbol{L}\lambda_\nu;$$

and

$$\begin{aligned}[L\lambda_\nu, vB] &= [\Pi\{L_q\lambda_\nu : q \in X\}, vB] \\ &= \Pi\{[L_q\lambda_\nu, vB] : q \in X\} \leqslant \Pi\{[L_q\lambda_\nu, v_q B] : q \in X\} \\ &\leqslant U,\end{aligned}$$

which finishes the proof of the theorem except for the uniqueness of $\boldsymbol{L}$: if there exists $\boldsymbol{L}'$, v' with the asserted properties, then

$$[L'\lambda_\nu, v'B] \leqslant L\lambda_\nu \quad \text{and} \quad [L\lambda_\nu, vB] \leqslant L'\lambda_\nu,$$

and lemma 4.2.15 shows that $L'\lambda_\nu \leqslant L\lambda_\nu \leqslant L'\lambda_\nu$, or $L\lambda_\nu = L'\lambda_\nu$ when $\boldsymbol{L} = \boldsymbol{L}'$ from (4.2.21).

The last lemma necessary to prove theorem 4.2.3 is the following.

Lemma. *Suppose $\Lambda_{\nu-1}$ has ascending chain condition. Let $\boldsymbol{L}$ be in $\Lambda_{\nu-1}$ with L contained in $\hat{A}$, and let v be a natural number. There exists a natural number $s = s(\boldsymbol{L}, v)$ such that if q in $L\lambda_\nu$ is uniform and involves more than s elements of the free generating set $\{z_1, z_2, \ldots\}$ then q is in $[L\lambda_\nu, vB]$.* (4.2.28)

Proof. The proof will be by induction on ν. If $\nu = 1$ then q in $\mathbf{L}\lambda_\nu$ can be written

$$q = \prod_{i=1}^{t} [y_1, z_1^{\delta_{i1}}, \ldots, z_u^{\delta_{iu}}]^{\alpha_i},$$

where $1 \leqslant \delta_{ij} \leqslant p-1$ for all i, j, and $(\delta_{i1}, \ldots, \delta_{iu})$ are distinct for distinct i. Employ (4.2.7) u times to deduce that

$$y_1^{\alpha_i} \in (\mathbf{L}\lambda_\nu)_\delta, \quad i \in \{1, \ldots, t\},$$

where $\delta = \sum_{j=1}^{u} \max_i \delta_{ij}$. Lemma 4.2.15 then yields that $y_1^{\alpha_i}$ belongs to $\mathbf{L}\lambda_\nu$ whence q belongs to $[\mathbf{L}\lambda_\nu, uB]$. Hence $s = v$ will do, and the proof of the first step is complete.

Assume, therefore, that ν is at least 2 and that the lemma is proved for $\nu - 1$. Associate with q the uniform †-biwords $q_1, \ldots, q_d$ of (4.2.13). By (4.2.13) and (4.2.15), $q_1, \ldots, q_d$ belongs to $\mathbf{L}\lambda_\nu$. Suppose that q_i involves s_i variables z_j, $i \in \{1, \ldots, d\}$. Let $\hat{\mathbf{L}}$ in $\Lambda_{\nu-2}$ be defined from $\mathbf{L}$ according to (4.2.27), and define

$$s(\mathbf{L}, v) = s(\hat{\mathbf{L}}, v(\mathbf{L}) + v) + v,$$

where $v(\mathbf{L})$ is defined as in (4.2.27), assuming inductively that s can be defined for $\nu - 1$.

Now by (4.2.13), q is in the normal closure of $q_1, \ldots, q_d$. Hence we may write

$$q = \prod_{j=1}^{t} [q_{i_j}, z_{k_{j1}}^{\alpha_{j1}}, \ldots, z_{k_{jr_j}}^{\alpha_{jr_j}}]^{\beta_j},$$

where $1 \leqslant \alpha_{jl} \leqslant p^\nu - 1$, all j, l. We may assume, by using suitable deletions that if q involves precisely the variables $y_1, z_1, \ldots, z_u$ (where $u \geqslant s(\mathbf{L}, v)$) then for each j, the set of variables z_w involved in q_{i_j} together with $z_{k_{j1}}, \ldots, z_{k_{jr_j}}$ is just $\{z_1, \ldots, z_u\}$. If for some j in $\{1, \ldots, t\}$, $s_{i_j} \leqslant s(\mathbf{L}, v) - v$ then $|\{k_{j1}, \ldots, k_{jr_j}\}| \geqslant v$ and therefore the commutator beginning with q_{i_j} belongs to $[\mathbf{L}\lambda_\nu, vB]$. If, on the other hand, $s_{i_j} > s(\mathbf{L}, v) - v$ for some j in $\{1, \ldots, t\}$, then $s_{i_j} > s(\hat{\mathbf{L}}, v(\mathbf{L}) + v)$; hence

$$\begin{aligned} q_{i_j} \xi_\nu^{-1} &\in [\hat{\mathbf{L}}\lambda_{\nu-1}, (v(\mathbf{L}) + v)\, \hat{B}] \\ &= [[\hat{\mathbf{L}}\lambda_{\nu-1}, v(\mathbf{L}) \hat{B}], v\hat{B}] \\ &\leqslant [\mathbf{L}, v\hat{B}] \end{aligned}$$

so that $q_{i_j} \in [\mathbf{L}, v\hat{B}]\, \lambda_\nu \leqslant [\mathbf{L}\lambda_\nu, vB]$. Clearly, then, the commutator beginning with this q_{i_j} belongs to $[\mathbf{L}\lambda_\nu, vB]$. Therefore q belongs to $[\mathbf{L}\lambda_\nu, vB]$.

Proof of (4.2.3). We use induction on ν, the result being obvious for $\nu = 0$. Suppose, therefore, that $\Lambda_{\nu-1}$ has ascending chain condition for some $\nu \geqslant 1$ and that $\mathbf{U}_1 \leqslant \mathbf{U}_2 \leqslant \ldots \leqslant \mathbf{U}_i \leqslant \ldots$ is an ascending chain in Λ_ν. Clearly the chain

$$U_1 \cap B \leqslant U_2 \cap B \leqslant \ldots \leqslant U_i \cap B \leqslant \ldots$$

terminates in a finite number of steps; hence it suffices to consider the chain of the $U_i \cap A$, or, without loss of generality, to assume $U_i \leqslant A$, $i \in I^+$. In this case (4.2.27) ensures that there exists to each i in I^+ a unique $\mathbf{L}_i$ in $\Lambda_{\nu-1}$ and an integer v_i such that

$$[L_i \lambda_\nu, v_i B] \leqslant U_i \leqslant L_i \lambda_\nu.$$

Now $i \leqslant j$ implies $\qquad [L_i \lambda_\nu, v_i B] \leqslant U_i \leqslant U_j \leqslant L_j \lambda_\nu$

and (4.2.15) and (4.2.26) give $L_i \leqslant L_j$. Under the inductive hypothesis it follows that there exists an integer m such that for $m \leqslant i$, $L_m = L_i$. Hence for $m \leqslant i$

$$[L_m \lambda_\nu, v_m B] \leqslant U_i \leqslant L_m \lambda_\nu.$$

By virtue of (4.2.28) there exists an integer $s_0 = s(\mathbf{L}_m, v_m)$ such that if q in $\mathbf{U}_i$ is uniform and involves more than s_0 variables z_j, then q belongs to $[L_m \lambda_\nu, v_m B]$. It follows that U_i can be determined, modulo $[L_m\lambda_\nu, v_m B]$, by bilaws involving at most $y_1, z_1, \ldots, z_{s_0}$. By theorems 2.1.2 and 1.5.4, $\boldsymbol{L}_m\lambda_\nu$ is finitely based, and therefore so is the sub-bigroup carried by $[L_m\lambda_\nu, v_m B]$; we may suppose the latter to have a basis involving t_0 variables z_j. Hence $\boldsymbol{U}_i$ $(m \leqslant i)$ is defined by laws involving at most s_0+t_0 variables z_j. It follows that the biverbal sub-bigroup lattice between the sub-bigroups carried by $[L_m\lambda_\nu, v_m B]$ and $L_m\lambda_\nu$ is isomorphic to the corresponding one in the free bigroup of rank $(1, s_0+t_0)$ of $\mathfrak{A}\circ\mathfrak{A}_{p^\nu}$. This is, however, a finitely generated metabelian group, and, by a well-known result of P. Hall (1954), has ascending chain condition on normal subgroups. This completes the proof of (4.2.3).

Remark. Now that (4.2.3) is proved the condition on $\Lambda_{\nu-1}$ in (4.2.27) and (4.2.28) is unnecessary.

Combining (4.2.3) and (4.1.18) we have

Theorem. *For all natural numbers m, n* $\Lambda(\mathfrak{A}_m\circ\mathfrak{A}_n)$ *has minimum condition.* (4.2.29)

(Note that, by varying slightly some of the proofs in § 4.1, we could strengthen theorem 4.2.29 to give minimum condition for the lattice $\Lambda(\mathfrak{A}\circ\mathfrak{A}_n)$; this was done in Bryce (1967) but is unnecessary here as we shall see.)

We are now in a position to prove the following theorem.

Theorem. *Let* $\mathscr{U}$ *be a sub-bivariety of* $\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^\beta}$ *which contains* $\mathfrak{E}\circ\mathfrak{A}_{p^\beta}$. *Then there exists a unique sub-bivariety* $\mathscr{L}$ *of* $\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^{\beta-1}}$ *containing* $\mathfrak{E}\circ\mathfrak{A}_{p^{\beta-1}}$ *and a nilpotent bivariety* $\mathscr{N}$ *such that*

$$\mathscr{U} = (\mathscr{L}(\mathfrak{E}\circ\mathfrak{A}_p)\wedge\mathfrak{A}\circ\mathfrak{A})\vee\mathscr{N}.$$

Moreover if $\mathscr{U}$ *is not nilpotent it is join-irreducible if and only if* $\mathscr{L}$ *is join-irreducible.* (4.2.30)

Proof. The lattice homomorphism $\lambda_\beta: \Lambda_{\beta-1}\to\Lambda_\beta$ defined in (4.2.20) induces a lattice homomorphism $\lambda^*_\beta: \Lambda(\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^{\beta-1}})\to\Lambda(\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^\beta})$ (by virtue of (1.4.14) and (1.5.4)) which is one-to-one by (4.2.21). Indeed all the properties of λ_β proved in (4.2.21) to (4.2.28) hold for λ^*_β (or rather for its dual). In particular, if $\mathfrak{E}\circ\mathfrak{A}_{p^\beta}\leqslant\mathscr{U}\leqslant\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^\beta}$ then there exists a unique $\mathscr{L}$ (by (4.2.27)) with $\mathfrak{E}\circ\mathfrak{A}_{p^{\beta-1}}\leqslant\mathscr{L}\leqslant\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^{\beta-1}}$ and an integer v such that

$$\mathscr{L}\lambda^*_\beta \leqslant \mathscr{U} \leqslant [\mathscr{L}\lambda^*_\beta, v\mathfrak{E}\circ\mathfrak{E}]. \tag{4.2.31}$$

Also it follows from (4.2.28) that for some suitably large natural number c,

$$[\mathscr{L}\lambda^*_\beta, v\mathfrak{E}\circ\mathfrak{E}] \leqslant \mathscr{L}\lambda^*_\beta\vee(\mathscr{N}_c\wedge\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^\beta}) \tag{4.2.32}$$

(here definitely using the finite exponent to give that a commutator of high weight is a uniform biword involving large numbers of variables; $\mathscr{N}_c$ is the variety of bigroups whose carriers have class at most c). The modular law applied to (4.2.31) and (4.2.32) then gives the first assertion of the theorem, since

$$\mathscr{L}\lambda^*_\beta = \mathscr{L}(\mathfrak{E}\circ\mathfrak{A}_p)\wedge\mathfrak{A}\circ\mathfrak{A}.$$

If $\mathscr{U}$ is join-irreducible then $\mathscr{U} = \mathscr{L}\lambda^*_\beta$ and since λ^*_β is an isomorphism, $\mathscr{L}$ must be join-irreducible. Conversely, suppose that $\mathfrak{E}\circ\mathfrak{A}_{p^{\beta-1}}\leqslant\mathscr{L}\leqslant\mathfrak{A}_{p^\alpha}\circ\mathfrak{A}_{p^{\beta-1}}$ and that $\mathscr{L}$ is join-irreducible whilst

$$\mathscr{L}\lambda^*_\beta = \mathscr{U}_1\vee\mathscr{U}_2.$$

At least one of $\mathscr{U}_1$, $\mathscr{U}_2$, say $\mathscr{U}_1$, must contain $\mathfrak{E}\circ\mathfrak{A}_{p^\beta}$; if $\mathscr{U}_2$ also does, then we may apply the first part of the theorem to deduce the existence of $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{N}'_1$, $\mathscr{N}''_1$ such that

$$\mathscr{L}\lambda^*_\beta = (\mathscr{L}_1\lambda^*_\beta\vee\mathscr{L}_2\lambda^*_\beta)\vee(\mathscr{N}'_1\vee\mathscr{N}''_1)$$

whence $\mathscr{L} = \mathscr{L}_1 \vee \mathscr{L}_2$ (by the uniqueness) and so $\mathscr{L} = \mathscr{L}_1$, say; and $\mathscr{L}\lambda^*_\beta \leqslant \mathscr{U}_1 \leqslant \mathscr{L}\lambda^*_\beta$ which proves what we want. If, on the other hand, $\mathscr{U}_2 \wedge \mathfrak{E} \circ \mathfrak{A}_{p\beta} = \mathfrak{E} \circ \mathfrak{A}_{p\gamma}$ with $0 \leqslant \gamma < \beta$ and $\mathscr{U}_2$ is not a subvariety of $\mathscr{U}_1$, what we have just shown proves that

$$\mathscr{L}\lambda^*_\beta = \mathscr{U}_2 \vee \mathfrak{E} \circ \mathfrak{A}_{p\beta}.$$

Now γ is non-zero or else $\mathscr{L} = \mathfrak{A}_{p\alpha'} \circ \mathfrak{E} \vee \mathfrak{E} \circ \mathfrak{A}_{p\beta-1}$; it follows that $\mathscr{L}\lambda^*_\beta$ has a bilaw $[y_1, z_1^{p\gamma}]$ and therefore that $\mathscr{L}$ has a bilaw $[y_1, z_1^{p\delta}]$ where $\beta - 1 > \delta = \gamma - 1 \geqslant 0$. This means that

$$\mathfrak{E} \circ \mathfrak{A}_{p\beta-1} \leqslant \mathscr{L} \leqslant \mathfrak{A}_{p\alpha} \circ \mathfrak{A}_{p\delta} \vee \mathfrak{E} \circ \mathfrak{A}_{p\beta-1}$$

and then, by modularity, that

$$\mathscr{L} = (\mathfrak{A}_{p\alpha} \circ \mathfrak{A}_{p\delta} \wedge \mathscr{L}) \vee \mathfrak{E} \circ \mathfrak{A}_{p\beta-1}.$$

The join-irreducibility of $\mathscr{L}$ and the fact that $\mathscr{U}_1$ does not contain $\mathscr{U}_2$ would now give that $\mathfrak{E} \circ \mathfrak{A}_{p\beta-1}$ is not contained in $\mathscr{L}$, a contradiction. Hence $\mathscr{U}_2$ is a subvariety of $\mathscr{U}_1$, and we are finished.

To sum up the results of this chapter on join-irreducibles:

THEOREM. *The non-nilpotent join-irreducibles in* $\Lambda(\mathfrak{A} \circ \mathfrak{A})$ *of finite exponent can be described (iteratively) as follows*:

(i) $\mathscr{L}(\mathfrak{E} \circ \mathfrak{A}_p) \wedge \mathfrak{A} \circ \mathfrak{A}$, *p prime*, $\mathscr{L}$ *join-irreducible of p-power exponent*, $\mathscr{L}\phi \neq \mathfrak{E} \circ \mathfrak{E}$;

(ii) $\mathscr{U}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A}$, $\mathscr{U}$ *join-irreducible of p-power exponent (p prime)*, $\mathscr{U}\phi \neq \mathfrak{E} \circ \mathfrak{E}$, $p \nmid t$, $t \in \{2, 3, \ldots\}$. (4.2.33)

4.3. *The bivarieties* $\mathfrak{A}_{p\alpha} \circ \mathfrak{A}_p$

The problem of determining all subvarieties of $\mathfrak{A}_m \circ \mathfrak{A}_n$ has been reduced (though with possible ambiguity) to the case when m, n are powers of the same prime. In general this case seems to be difficult, and (4.2.30) is the best we can do. Only in the case $\beta = 1$ do we get a complete picture. First we prove two lemmas similar to (4.2.7).

LEMMA. *If, in the notation of* (4.2.2), $a_0, \ldots, a_{p-1}$ *are fixed elements of* A, *and if* $\mathbf{U}$ *is normal in* $\mathbf{W}_\nu$ *such that for all* b *in* B

$$\rho = \prod_{i=0}^{p-1} [a_i, ib] \in U,$$

then $a_i \in U_i$ $(0 \leqslant i \leqslant p-1)$. (4.3.1)

Proof. Using the identity (0.2.1) we may express ρ as

$$\rho = \prod_{i=0}^{p-1} [a'_i, b^i] \in U,$$

where each a'_i is a linear combination of $a_i, \ldots, a_{p-1}$, and $a'_{p-1} = a_{p-1}$. From (4.2.7) we deduce that a_{p-1} is in U_{p-1}, whence

$$\prod_{i=0}^{p-2} [a_i, ib] \in U.$$

An easy induction is indicated to finish the proof, and the details are omitted.

LEMMA. *Define* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s)$ *where* $0 \leqslant \mu_i \leqslant p-1$ *for all* i. *If* $a(\boldsymbol{\mu})$ *are fixed elements of* A, *and if for all* $b_1, \ldots, b_s$ *in* B

$$\prod_{\boldsymbol{\mu}} [a(\boldsymbol{\mu}), \mu_1 b_1, \ldots, \mu_s b_s] \in U$$

(*where* $\mathbf{U}$ *is normal in* $\mathbf{W}_\nu$), *then* $a(\boldsymbol{\mu}) \in U_\tau$ *where* $\tau = \mu_1 + \ldots + \mu_s$. (4.3.2)

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Proof. We proceed by induction on s, the case $s = 1$ being covered by the last lemma. For $i \in \{0, \ldots, p-1\}$ write

$$a_i = \prod_{\mu_s = i} [a(\boldsymbol{\mu}), \mu_1 b_1, \ldots, \mu_{s-1} b_{s-1}];$$

then $\Pi_{i=0}^{p-1} [a_i, ib]$ belongs to U for all b in B. Hence, by (4.3.1), a_i belongs to U_i, $i \in \{0, \ldots, p-1\}$. Now $\mu_s = \mu'_s$ implies $(\mu_1, \ldots, \mu_{s-1})$ is different from $(\mu'_1, \ldots, \mu'_{s-1})$ if $\boldsymbol{\mu}$ is different from $\boldsymbol{\mu}'$. We may then, by induction, assume that

$$a(\boldsymbol{\mu}) \in (U_i)_j,$$

where $j = \mu_1 + \ldots + \mu_{s-1}$. That is, $a(\boldsymbol{\mu})$ is in U_τ, $\tau = i + j = \mu_1 + \ldots + \mu_s$, for each $\boldsymbol{\mu}$ as required.

Before beginning the statement and proof of our main results in this section, we introduce the following notation. Write $\boldsymbol{X}_\alpha$ for the split-free bigroup of rank $(1, \infty)$ in $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p$ on the split-free generating set $\{y_1\} \cup \{z_1, z_2, \ldots\}$. It is clear from (4.2.1) that the lattice of normal, fully invariant sub-bigroups of $\boldsymbol{X}_\alpha$ is dually isomorphic to $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p)$. Write (d, σ) for the fully invariant closure of $[y_1, z_1, \ldots, z_d]^{p^\sigma}$ in $\boldsymbol{X}_\alpha$; abusing convention, then

Notation. *For* $d \geqslant 0$, $\sigma \in \{0, 1, \ldots, \alpha - 1\}$

$$(d, \sigma) = cl\{[y_1, z_1, \ldots, z_d]^{p^\sigma}\}. \tag{4.3.3}$$

Theorem. *Every fully invariant sub-bigroup of* $\boldsymbol{X}_\alpha$ *contained in* $A_1(\boldsymbol{X}_\alpha)$ *can be written as a product of finitely many* (d, σ)*'s.* (4.3.4)

Proof. From (2.2.4), every fully invariant $\boldsymbol{U}$ contained in $A_1(\boldsymbol{X}_\alpha)$ is the closure of uniform biwords of the type

$$q = \prod_{i=1}^{t} [y_1, \mu_{i1} z_1, \ldots, \mu_{is} z_s]^{\alpha_i},$$

where $1 \leqslant \mu_{ij} \leqslant p-1$, $1 \leqslant \alpha_i \leqslant p^\alpha - 1$, all i, j, and where $i \neq j$ implies $(\mu_{i1}, \ldots, \mu_{is}) \neq (\mu_{j1}, \ldots, \mu_{js})$. Lemma 4.3.2 gives that

$$y_1^{\alpha_i} \in (cl\{q\})_{\tau_i}, \quad \tau_i = \mu_{i1} + \ldots + \mu_{is}.$$

Clearly, then q is equivalent to a finite set of (d, σ)'s and therefore so is $\boldsymbol{U}$.

With this theorem we can in fact determine all sub-bivarieties of $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p$; however, we have as yet no way of knowing when two different sets of (d, σ)'s determine different sub-bivarieties. We take up this problem now.

Theorem. *The commutators*

$$[y_1, \mu_1 z_1, \ldots, u_r z_r],$$

$r \geqslant 0$, $0 \leqslant \mu_i \leqslant p-1$ *for* $i \in \{1, \ldots, r\}$ *and* $\mu_r > 0$, *form a basis for* $A_1(\boldsymbol{X}_\alpha)$. *If* $d > 0$, *then a basis for* (d, σ) *is the set of all* b^{p^τ}, *where* b *is a basic commutator of weight at least* 2 *and where* τ *is minimal with respect to* $\sigma \leqslant \tau$ *and* $\mathrm{wt}\, b + (\tau - \sigma)(p-1) \geqslant d+1$; *the set* $\{b^{p^\sigma}\colon b \text{ basic}\}$ *is a basis for* $(0, \sigma)$. (4.3.5)

Proof. The set of commutators of the type described certainly generate $A_1(\boldsymbol{X}_\alpha)$: the only thing to check is that, using the identity

$$[y_1, pz_1] = \prod_{i=1}^{p-1} [y_1, iz_1]^{-\binom{p}{i}}$$

we can remove p or more repetitions of any variable z_j, replacing the offending commutator by a product of commutators each of which has fewer than p occurrences of z_j. That these commutators with few repetitions are basic follows from (4.3.2); for, if

$$\prod_{i=1}^{t} [y_1, \mu_{i1} z_1, \ldots, \mu_{is_i} z_{s_i}]^{\alpha_i} = 1,$$

where $(\mu_{i1}, \ldots, \mu_{is_i})$ is different from $(\mu_{j1}, \ldots, \mu_{js_j})$ whenever i is not j, and where $0 \leqslant \mu_{il} \leqslant p-1$, $\mu_{is_i} > 0$ for all i in $\{1, \ldots, t\}$ and l in $\{1, \ldots, s_i\}$, then, if $s = \max\{s_i : 1 \leqslant i \leqslant t\}$ we have, by defining $\mu_{il} = 0$ for $s_i < l \leqslant s$ where necessary, that

$$\prod_{i=1}^{t} [y_1, \mu_{i1} z_1, \ldots, \mu_{is} z_s]^{\alpha_i} = 1,$$

with $(\mu_{i1}, \ldots, \mu_{is})$ different from $(\mu_{j1}, \ldots, \mu_{js})$ when i is not j. We may therefore apply lemma 4.3.2 to deduce for each $i \in \{1, \ldots, t\}$, that

$$[y_1, z_1, \ldots, z_\tau]^{\alpha_i} = 1,$$

where $\tau = \mu_{i1} + \ldots + \mu_{is}$; this would then be a bilaw in $\boldsymbol{X}_\alpha$, and therefore p^α divides α_i. For if not, then $[y_1^{p^{\alpha-1}}, z_1, \ldots, z_\tau] = 1$ and therefore $[y_1, z_1, \ldots, z_\tau]$ is a bilaw in $C_p \operatorname{wr} C_p^\tau$, which is not true (see Liebeck 1962). Hence p^α divides α_i for all i, and this shows that the set of commutators $[y_1, \mu_1 z_1, \ldots, \mu_r z_r]$ with $r \geqslant 0$, $0 \leqslant \mu_i \leqslant p-1$ and $\mu_r > 0$ is a basis for $A_1(\boldsymbol{X}_\alpha)$.

It is quite clear that the set $\{b^{p^\sigma} : b \text{ basic}\}$ is a basis for $(0, \sigma)$, but the remaining assertion of the theorem requires proof. The crucial point is the following result.

LEMMA. *$(e, \tau) \leqslant (d, \sigma)$ if and only if $\sigma \leqslant \tau$, $d = 0$ if $e = 0$ and $d \leqslant e + (\tau - \sigma)(p-1)$ if $e > 0$.* (4.3.6)

Proof. The first part is easy: if $(e, \tau) \leqslant (d, \sigma)$ then $[y_1, z_1, \ldots, z_e]^{p^\tau}$ can be written as a product of p^σ-th powers, and hence, if $\sigma > \tau$, $[y_1, z_1, \ldots, z_e]^{p^{\alpha-1}} = 1$ which, as we have observed, is impossible. Also if $e = 0$ and $d > 0$, then $y_1^{p^\tau}$ can be written as a product of commutators all involving at least one z_j; then by deleting z_j for all j, we have $y_1^{p^\tau} = 1$ which is a contradiction.

Suppose therefore, that $e > 0$ and $\sigma \leqslant \tau$. Then we prove that

$$(e, \tau) \leqslant (e + (\tau - \sigma)(p-1), \sigma) \tag{4.3.7}$$

and

$$(e, \tau) \nleqslant (e + (\tau - \sigma)(p-1) + 1, \sigma). \tag{4.3.8}$$

Consider the identity

$$[y_1, z_1, \ldots, z_{e+r}]^p = \prod_{i=2}^{p} [y_1, z_1, \ldots, z_{e+r-1}, iz_{e+r}]^{-\binom{p}{i}};$$

from this one deduces that for $r \leqslant p-2$

$$(e+r+1, 1) \leqslant (e+p-1, 0)$$

implies

$$(e+r, 1) \leqslant (e+p-1, 0)$$

and therefore, by downward induction on r, $(e, 1) \leqslant (e+p-1, 0)$. This then gives by induction on $\tau - \sigma$ ((4.3.7) is trivially true if $\tau = \sigma$),

$$\begin{aligned}(e, \tau) &= (e, 1)^{p^{\tau-1}} \leqslant (e+p-1, 0)^{p^{\tau-1}} \\ &= (e+p-1, \tau-1) \leqslant (e+p-1+(\tau-1-\sigma)(p-1), \sigma) \\ &= (e+(\tau-\sigma)(p-1), \sigma).\end{aligned}$$

This proves (4.3.7). The proof of (4.3.8) is more difficult, and uses the next two lemmas.

LEMMA. *If $m > 0$ and*

$$[y_1, mz_1] = \prod_{i=1}^{p-1} [y_1, iz_1]^{\delta(m, i)}$$

and if $m = p + (\mu - 1)(p-1) + r$, $0 \leqslant r < p-1$, $0 \leqslant \mu$, then

(i) *$\mu = 0$ implies $\delta(m, i) = 1, 0$ according as $m = i$ or $m \neq i$;*

(ii) *$r = 0$ implies $p^\mu | \delta(m, i)$, $1 \leqslant i \leqslant p-1$;*

(iii) *$\mu r \geqslant 1$ implies $p^{\mu+1} | \delta(m, i)$, $1 \leqslant i \leqslant r$ and $p^\mu | \delta(m, i), r+1 \leqslant i \leqslant p-1$.* (4.3.9)

Proof. Clearly (i) is a consequence of the uniqueness already proved in (4.3.5). For $\mu = 1, r = 0$, (ii) is easily seen to be true. Suppose that the lemma has been proved for some m with $m \geqslant p$. Then

$$\begin{aligned}[y_1, (m+1)\, z_1] &= [y_1, z_1, mz_1] \\ &= \prod_{i=1}^{p-1} [y_1, z_1, iz_1]^{\delta(m,\, i)} \\ &= \prod_{i=1}^{p-2} [y_1, (i+1)\, z_1]^{\delta(m,\, i)} \prod_{i=1}^{p-1} [y_1, iz_1]^{-\binom{p}{i}\,\delta(m,\, p-1)}\end{aligned}$$

and so, by the uniqueness from (4.3.5),

$$\delta(m+1, i) = \delta(m, i-1) - \tbinom{p}{i}\delta(m, p-1), \quad 2 \leqslant i \leqslant p-1, \quad \delta(m+1, 1) = -p\delta(m, p-1).$$

By assumption $p^{\mu+1} | \delta(m, i)$, $i \leqslant r$ and $p^{\mu} | \delta(m, 1)$, $r < i$, whence the proof may be completed.

LEMMA. *If* $m_1, \ldots, m_d \geqslant 1$ *and*

$$[y_1, m_1 z_1, \ldots, m_d z_d] = \prod_{\boldsymbol{i}} [y_1, i_1 z_1, \ldots, i_d z_d]^{\beta(\boldsymbol{i})}$$

where $\boldsymbol{i} = (i_1, \ldots, i_d)$ *with* $1 \leqslant i_j \leqslant p-1$, *then* $m_1 + \ldots + m_d \geqslant d + \tau(p-1) + 1$ *implies* $p^{\tau+1} | \beta(1, \ldots, 1)$. (4.3.10)

Proof. With $d = 1$ we have $d + \tau(p-1) + 1 = p + (\tau - 1)(p-1) + 1$ and lemma 4.3.9 applies. We use this as a starting point for induction on d. Suppose $m_d = \phi(p-1) + \rho \geqslant 1$, $0 \leqslant \rho < p-1$, $0 \leqslant \phi$. Then

$$m_1 + \ldots + m_{d-1} \geqslant (d-1) + (\tau - \phi)(p-1) - (\rho - 2).$$

Now if $[y_1, m_1 z_1, \ldots, m_{d-1} z_{d-1}] = \Pi_{\boldsymbol{i}} [y_1, i_1 z_1, \ldots, i_{d-1} z_{d-1}]^{\gamma(\boldsymbol{i})}$, then we may assume inductively that

$$\begin{aligned} p^{\tau-\phi+1} | \gamma(1, \ldots, 1) \quad &\text{if} \quad \rho \leqslant 1, \\ p^{\tau-\phi} | \gamma(1, \ldots, 1) \quad &\text{if} \quad 1 < \rho. \end{aligned}$$

Also from (4.3.5),

$$\beta(1, \ldots, 1) = \delta(m_d, 1)\, \gamma(1, \ldots, 1);$$

and

$$\begin{aligned} p^{\phi+1} | \delta(m_d, 1) \quad &\text{if} \quad 1 < \rho, \\ p^{\phi} | \delta(m_d, 1) \quad &\text{if} \quad \rho \leqslant 1. \end{aligned}$$

In any case, $p^{\tau+1} | \beta(1, \ldots, 1)$ as required.

Proof of (4.3.8). If $(e, \tau) \leqslant (e + (\tau - \sigma)(p-1) + 1, \sigma)$, then

$$[y_1, z_1, \ldots, z_e]^{p^\tau} = \Pi_{\boldsymbol{j}} [y_1, j_1 z_1, \ldots, j_e z_e]^{p^\sigma \beta'(\boldsymbol{j})} \tag{$*$}$$

where $j_1 + \ldots + j_e \geqslant e + (\tau - \sigma)(p-1) + 1$. Now $(*)$ can be rewritten by replacing each $[y_1, j_1 z_1, \ldots, j_e z_e]$ by a product of powers of basic commutators. Then, using the uniqueness from (4.3.5),

$$[y_1, z_1, \ldots, z_e]^{p^\tau} = \Pi_{\boldsymbol{j}} [y_1, z_1, \ldots, z_e]^{p^\sigma \beta(\boldsymbol{j})},$$

where for each $\boldsymbol{j}$, $p^{\tau-\sigma+1} | \beta(\boldsymbol{j})$ by (4.3.10). Hence

$$p^\tau = p^\sigma \Sigma \beta(\boldsymbol{j}),$$

and since the right-hand side of this equation is divisible by $p^{\tau+1}$ we have a contradiction. This completes the proof of (4.3.8).

Proof of (4.3.5). If $d > 0$ and $b_1, \ldots, b_t$ are distinct basic commutators such that

$$b_1^{\beta_1} \ldots b_t^{\beta_t} \in (d, \sigma),$$

then, from (4.3.2), if b_i has weight e_i+1, and p^{τ_i} is the largest power of p dividing β_i

$$(e_i, \tau_i) \leqslant (d, \sigma)$$

whence, from the part of (4.3.5) already proved, and (4.3.6),

$$\sigma \leqslant \tau_i, \quad e_i > 0, \quad d \leqslant e_i + (\tau_i - \sigma)(p-1).$$

This completes the proof of (4.3.5).

The main result of this section can now be stated. As the proof is of a routine nature using theorem 4.3.5 we will omit most of the details.

THEOREM. *Every non-trivial normal, fully invariant sub-bigroup* $\boldsymbol{U}$ *of* $\boldsymbol{X}_\alpha$ *can be written uniquely as*

$$\boldsymbol{U} = A_2(\boldsymbol{X}_\alpha)^\epsilon \,.\, (d_\sigma, \sigma) \ldots (d_{\alpha-1}, \alpha-1)$$

where $\epsilon = 0, 1$ (*according as* $z_1 \notin \boldsymbol{U}$ *or* $z_1 \in \boldsymbol{U}$) *and*

(i) $\epsilon = 1$ *implies* $\sigma = 0$, $d_\sigma \leqslant 1$;

(ii) *if* $\phi \in \{\sigma, \ldots, \alpha-2\}$ *then*

$$d_{\phi+1} \begin{cases} \leqslant d_\phi - p + 1, & \text{if } \; p \leqslant d_\phi, \\ \leqslant 1, & \text{if } \; 1 \leqslant d_\phi \leqslant p-1, \\ = 0, & \text{if } \; 0 = d_\phi. \end{cases} \tag{4.3.11}$$

Proof. Theorem 4.3.4 ensures that every non-trivial $\boldsymbol{U}$ can be written as a join as indicated; if $z_1 \in \boldsymbol{U}$ then $[y_1, z_1] \in \boldsymbol{U}$ and hence $\boldsymbol{U}$ contains $(1, 0)$.

Let σ be the smallest element of $\{0, \ldots, \alpha-1\}$ for which (d, σ) is contained in $\boldsymbol{U}$ for some integer d, and let d_τ be the smallest integer such that (d_τ, τ) is contained in $\boldsymbol{U}$ for $\sigma \leqslant \tau \leqslant \alpha-1$. Since by (4.3.6)

$$(d, \tau+1) \leqslant (d+p-1, \tau)$$

for $d > 0$, we have that $d_\tau \geqslant p$ implies $d_{\tau+1} \leqslant d_\tau - p + 1$. If $1 \leqslant d_\tau \leqslant p-1$ then for all $d > 0$

$$(d, \tau+1) \leqslant (d+p-1, \tau) \leqslant (d_\tau, \tau) \leqslant U,$$

hence $d_{\tau+1} \leqslant 1$. If $d_\tau = 0$ for some $\tau \in \{\sigma, \ldots, \alpha-2\}$ then clearly $d_{\tau+1} = \ldots = d_{\alpha-1} = 0$. This establishes the existence of such a join decomposition for $\boldsymbol{U}$.

The uniqueness is a consequence of the next lemma, whose proof we omit.

LEMMA. *If* $(d, \tau) \leqslant (d_\sigma, \sigma) \ldots (d_{\alpha-1}, \alpha-1)$ *where* $d_\sigma, \ldots, d_{\alpha-1}$ *satisfy the condition* (ii) *of* (4.3.11), *then* $\sigma \leqslant \tau$ *and* $d_\tau \leqslant d$. (4.3.12)

COROLLARY. *Let* $J = \{0, 1, \ldots, i, \ldots\} \cup \{\infty\}$, (4.3.13)
and $T = \{0, 1\}$ *have their natural orders, then the lattice*

$$T \times J^\alpha$$

embeds $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p)$. *In particular* $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p)$ *is distributive.*

The details of proof are routine and we omit them.

COROLLARY. *Theorems* 4.3.11 *and* 4.1.8 *afford a complete description of* $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ *if m is nearly prime to n. In particular* $\Lambda(\mathfrak{A}_m \circ \mathfrak{A}_n)$ *is distributive in such cases.* (4.3.14)

COROLLARY. *The join-irreducibles in* $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p)$ *are*:

(i) *non-nilpotent*: $\mathfrak{A}_{p^\sigma} \circ \mathfrak{A}_p$, $\sigma \in \{1, \ldots, \alpha\}$;

(ii) *nilpotent*: $\mathfrak{E} \circ \mathfrak{A}_p$, $\mathfrak{A}_{p^\sigma} \circ \mathfrak{E}$, $\sigma \in \{0, 1, \ldots, \alpha\}$;

$$\mathfrak{A}_{p^\sigma} \circ \mathfrak{A}_p \wedge \mathcal{N}_c, \quad \sigma \in \{1, \ldots, \alpha\}, \quad c \geqslant (\sigma-1)(p-1)+2. \tag{4.3.15}$$

Proof. It follows from (4.2.33) immediately (and, of course, from (4.3.11), with more trouble) that the non-nilpotent join-irreducibles in $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_p)$ are as stated; hence we may concentrate on nilpotent ones $\mathscr{U}$ that contains $\mathfrak{E} \circ \mathfrak{A}_p$; and we may as well assume further that $\mathfrak{A}_{p^\alpha} \circ \mathfrak{E} \leqslant \mathscr{U}$. Then (4.3.11) yields that

$$U = (d_0, 0)\,(d_1, 1) \ldots (d_{\alpha-1}, \alpha-1), \tag{4.3.16}$$

where the d's satisfy condition (ii) of theorem 4.3.11 with $d_{\alpha-1} \geqslant 1$. Moreover, for all $\sigma \in \{0, \ldots, \alpha-2\}$,

$$d_\sigma - d_{\sigma+1} = p-1.$$

For, if not, let τ be the first element of $\{0, \ldots, \alpha-2\}$ for which $d_\tau - d_{\tau+1}$ is different from $p-1$; then if $d_\tau - d_{\tau+1}$ is greater than $p-1$ and we write

$$U_1 = (d_0-1, 0) \ldots (d_\tau - 1, \tau)\,(d_{\tau+1}, \tau+1) \ldots (d_{\alpha-1}, \alpha-1),$$

and

$$U_2 = (d_0, 0) \ldots (d_\tau, \tau)\,(d_{\tau+1}-1, \tau+1) \ldots (d_{\alpha-1}-1, \alpha-1),$$

these are expressions satisfying condition (ii) of (4.3.11) with

$$U_1 \neq U \neq U_2 \quad \text{and} \quad U = U_1 \cap U_2,$$

contrary to the assumption that $\mathscr{U}$ is join-irreducible; if $0 < d_\tau - d_{\tau+1} < p-1$ we conclude that $d_{\tau+1} = 1$ and, by writing

$$U_1^* = (d_0, 0) \ldots (d_{\tau-1}, \tau-1)\,(1, \tau)\,(1, \tau+1) \ldots (1, \alpha-1)$$

and

$$U_2^* = (d_0, 0) \ldots (d_{\tau-1}, \tau-1)\,(d_\tau, \tau)\,(0, \tau+1) \ldots (0, \alpha-1),$$

we see that these are expressions satisfying condition (ii) of (4.3.11) and that

$$U_1^* \neq U \neq U_2^* \quad \text{and} \quad U = U_1^* \cap U_2^*$$

again contradicting the join-irreducibility of $\mathscr{U}$; and finally if $d_\tau = d_{\tau+1}$ (= 1 of course) then we put

$$U_1^{**} = (d_0-1, 0) \ldots (d_{\tau-1}-1, \tau-1)\,(1, \tau)\,(1, \tau+1) \ldots (1, \alpha-1)$$

and

$$U_2^{**} = (d_0, 0) \ldots (d_{\tau-1}, \tau-1)\,(0, \tau) \ldots (0, \alpha-1),$$

so that

$$U_1^{**} \neq U \neq U_2^{**} \quad \text{and} \quad U = U_1^{**} \cap U_2^{**},$$

again a contradiction.

Hence if $\mathscr{U}$ is join-irreducible then the decomposition (4.3.16) is 'redundant'; that is $U = (d_0, 0)$ with

$$d_0 = (\alpha-1)(p-1) + d_{\alpha-1} \geqslant (\alpha-1)(p-1)+1.$$

It is easy to verify that $(d, 0)$ is join-irreducible if $d \geqslant (\alpha-1)(p-1)+2$ and not when $d = (\alpha-1)(p-1)+1$. The proof of (4.3.15) is therefore complete.

4.4. *The bivarieties* $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathscr{N}_c$

In this section we give a classification of another class of bivarieties, and produce an example of a non-distributive bivariety lattice. First note the following:

LEMMA. *A bigroup* $\mathbf{G}$ *in* $\mathfrak{A} \circ \mathfrak{A}$ *has the bilaw* $[y_1, z_1, \ldots, z_d]^m$ *if and only if* G *has the law*

$$[x_1, x_2, \ldots, x_{d+1}]^m. \tag{4.4.1}$$

Proof. Now G has the law $[x_1, x_2, \ldots, x_{d+1}]^m$ if and only if $\mathbf{G}$ has the bilaw $[y_1 z_1, \ldots, y_{d+1} z_{d+1}]^m$. Modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$ we have

$$[y_1 z_1, \ldots, y_{d+1} z_{d+1}] = [y_1, z_2, \ldots, z_{d+1}]^{z_1} [z_1, y_2, z_3, \ldots, z_{d+1}]^{z_2}$$

and therefore $[y_1 z_1, \ldots, y_{d+1} z_{d+1}]^m$ is equivalent, modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$, to $[y_1, z_1, \ldots, z_d]^m$.

Note that, in particular, G has class c if and only if $\mathbf{G}$ has the bilaw $[y_1, z_1, \ldots, z_c]$.

NOTATION. *Denote by $\mathcal{N}_c$ the variety of all bigroups whose carriers have class at most c.* (4.4.2)

NOTATION. *Let $\mathbf{Y}_\alpha$ be the split-free bigroup of rank $(1, \infty)$ in $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathcal{N}_p$, and again abuse convention by writing (d, σ) for the normal fully invariant closure of $[y_1, z_1, \ldots, z_d]^{p^\sigma}$ in $\mathbf{Y}_\alpha$, $d \in \{0, \ldots, p-1\}$, $\sigma \in \{0, \ldots, \alpha-1\}$.* (4.4.3)

THEOREM. *Every non-trivial normal, fully invariant sub-bigroup $\mathbf{U}$ of $\mathbf{Y}_\alpha$ can be written uniquely as*

$$\mathbf{U} = A_2(\mathbf{Y}_\alpha)^{p^\gamma} (d_\sigma, \sigma) \ldots (d_{\alpha-1}, \alpha-1)$$

where $\gamma \in \{0, \ldots, \alpha\}$, $\sigma \in \{0, \ldots, \alpha-1\}$, $p-1 \geqslant d_\sigma \geqslant \ldots \geqslant d_{\alpha-1} \geqslant 0$, and if $\gamma < \alpha$ then $\sigma \leqslant \gamma$ and $d_\gamma \leqslant 1$. $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathcal{N}_p)$ is distributive. (4.4.4)

Proof. That every $\mathbf{U}$ has a decomposition of this form follows from (2.2.4) and (4.3.2); choose σ as the smallest element of $\{0, \ldots, \alpha-1\}$ for which there exists d in $\{0, \ldots, p-1\}$ such that (d, σ) is a sub-bigroup of $\mathbf{U}$, then choose d_τ as the smallest d for which (d, τ) is a sub-bigroup of $\mathbf{U}$, $\sigma \leqslant \tau \leqslant \alpha-1$. Clearly then $d_\sigma \geqslant \ldots \geqslant d_{\alpha-1}$. The rest of the proof will follow easily from the next lemma which will also prove useful again in this section; we omit its proof.

LEMMA. *The carrier of the split-free bigroup of rank $(1, 1)$ in $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathcal{N}_{p+1}$ (where $\alpha > 1$) can be presented on the generators $a_0, \ldots, a_p, b$ subject to the defining relations*

$$a_0^{p^\alpha} = \ldots = a_{p-1}^{p^\alpha} = a_p^{p^{\alpha-1}} = b^{p^\alpha} = [a_i, a_j] = 1, \quad 0 \leqslant i, j \leqslant p,$$
$$a_i^b = a_i a_{i+1}, \quad a_p^b = a_p, \quad 0 \leqslant i \leqslant p-1. \qquad (4.4.5)$$

Return to the proof of (4.4.4). If (d, τ) is contained in $\mathbf{U}$ then (d, τ) is contained in $(d_\sigma, \sigma) \ldots (d_{\alpha-1}, \alpha-1)$ and therefore

$$\begin{aligned}(d, \alpha-1) &\leqslant (d_\sigma, \alpha-\tau-1+\sigma) \ldots (d_\tau, \alpha-1)\\ &\leqslant (d_\tau, \alpha-\tau-1+\sigma) \ldots (d_\tau, \alpha-1)\\ &= (d_\tau, \alpha-\tau-1+\sigma).\end{aligned}$$

However lemma 4.4.5 yields that, even in the free bigroup of rank $(1, 1)$ in $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathcal{N}_p$ (with $\alpha > 1$) this can happen only if $d \geqslant d_\tau$, $\alpha-1 \geqslant \alpha-\tau-1+\sigma$; that is, $d \geqslant d_\tau$ and $\tau \geqslant \sigma$, whence $(d, \tau) \leqslant (d_\tau, \tau)$. Since γ is quite clearly unique, we have shown that this expression for $\mathbf{U}$ is unique; it only remains to remark, that $z_1^{p^\gamma} \in \mathbf{U}$ implies $[y_1, z_1^{p^\gamma}] \in \mathbf{U}$ and that $[y_1, z_1^{p^\gamma}]$ and $[y_1, z_1]^{p^\gamma}$ are equivalent modulo the bilaws of $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathcal{N}_p$, from (4.4.5). As the case $\alpha = 1$ is covered by (4.3.11), this completes the proof of (4.4.4).

COROLLARY. *The join-irreducibles in $\Lambda(\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}_{p^\alpha} \wedge \mathcal{N}_p)$ are:*

$$\mathfrak{A}_{p^\sigma} \circ \mathfrak{E}, \quad \mathfrak{E} \circ \mathfrak{A}_{p^\sigma}, \quad \mathfrak{A}_{p^\tau} \circ \mathfrak{A}_{p^\tau} \wedge \mathcal{N}_c,$$

where $\sigma \in \{0, \ldots, \alpha\}$, $\tau \in \{1, \ldots, \alpha\}$ and $c \in \{2, \ldots, p\}$. (4.4.6)

COROLLARY. *Let $\mathscr{B}$ be a subvariety of $\mathfrak{A}_m \circ \mathfrak{A}_n$ in which the carriers of bigroups which are p-groups have class at most p. Then (4.4.4) and (4.1.8) provide a complete description of $\Lambda(\mathscr{B})$; in particular $\Lambda(\mathscr{B})$ is distributive.* (4.4.7)

THEOREM.‡ *$(\mathfrak{A}_{p^2}\circ\mathfrak{A}_{p^2}\wedge\mathcal{N}_{p+1})$ is not distributive.* (4.4.8)

Proof. We show that in the split-free bigroup of rank (1, 1) in $\mathfrak{A}_{p^2}\circ\mathfrak{A}_{p^2}\wedge\mathcal{N}_{p+1}$, there exist normal, fully invariant sub-bigroups V_1, V_2, V_3 which are pairwise incomparable and whose pairwise joins and intersections are respectively equal. Let V_1, V_2, V_3 be determined by the bilaws

$$[y_1, z_1]^p, \quad [y_1, z_1^p], \quad [y_1, pz_1]$$

respectively, and let V be determined by $[y_1, 2z_1]^p$. In the notation of (4.4.5) it is clear that

$$V = \langle a_2^p, \dots, a_{p-1}^p\rangle,$$
$$V_1 = \langle a_1^p, V\rangle, \quad V_3 = \langle a_p, V\rangle.$$

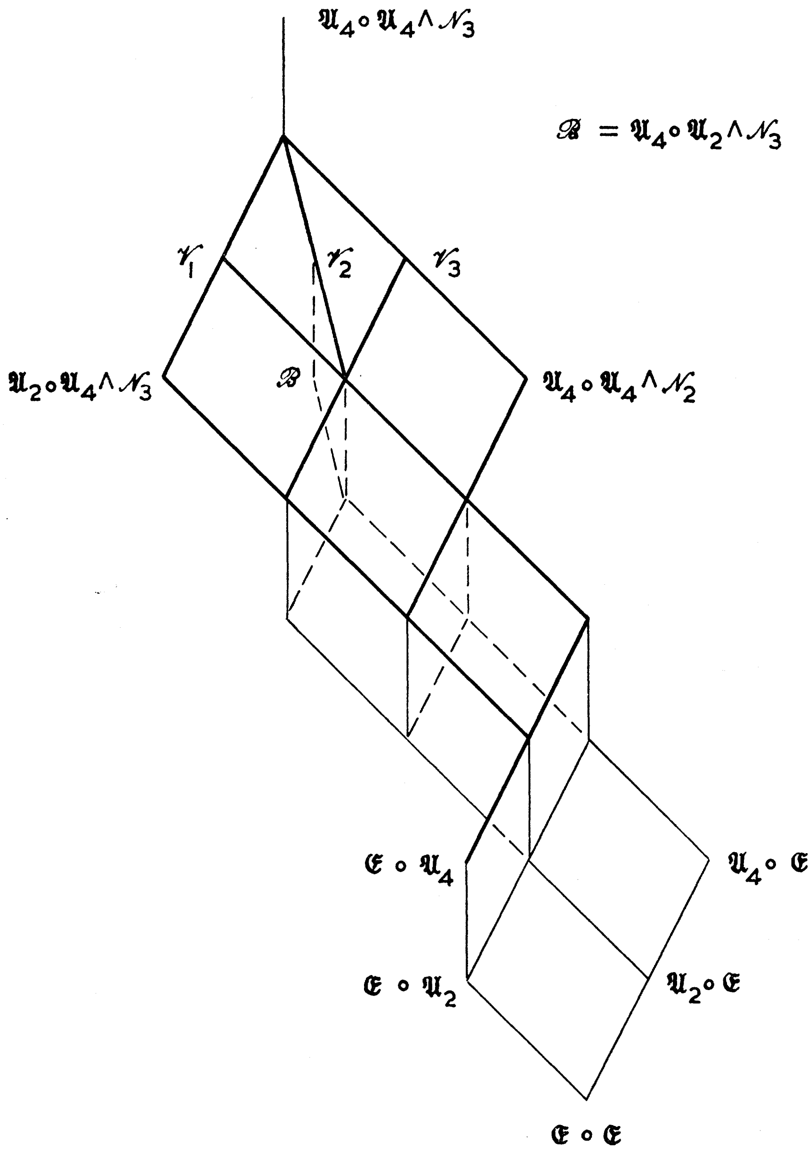

FIGURE 1

‡ Contrast $\Lambda(\mathfrak{A}\mathfrak{A}\wedge\mathfrak{N}_{p+1})$ which is distributive (W. Brisley, unpublished). A routine modification of the present result shows that $\Lambda(\mathfrak{A}_{p^2}\mathfrak{A}_{p^2}\wedge\mathfrak{N}_{p+2})$ is not distributive.

Also since
$$[a_0, b^{kp}] = \prod_{i=1}^{p} [a_0, ib]^{\binom{kp}{i}} = a_1^{kp} a_p^{\binom{kp}{p}}$$
$$= (a_1^p a_p)^k$$

modulo V $\left(\text{using the fact that } \binom{kp}{p} \equiv k (\operatorname{mod} p)\right)$, we have that
$$V_2 = \langle a_1^p a_p, V \rangle.$$

Hence (4.4.5) yields that $V_1V_2 = V_2V_3 = V_3V_1 = \langle a_1^p, a_p, V\rangle$, and $V_1 \cap V_2 = V_2 \cap V_3 = V_3 \cap V_1 = V$; and clearly V_1, V_2, V_3, V are all distinct. This completes the proof of (4.4.8); a picture of the lattice $\Lambda(\mathfrak{A}_4 \circ \mathfrak{A}_4 \wedge \mathscr{N}_3)$ is drawn by way of illustration (figure 1), but it is not here verified that it has this precise form.

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Chapter 5. Subvarieties of $\mathfrak{A} \circ \mathfrak{A}$ of infinite exponent

Theorem 4.2.33 gives a description of the join-irreducibles in $\Lambda(\mathfrak{A} \circ \mathfrak{A})$ of finite exponent in terms of the nilpotent ones. For infinite exponent subvarieties of $\mathfrak{A} \circ \mathfrak{A}$ we here obtain a canonic decomposition which enables us to write down all the infinite exponent join-irreducibles (for example see (5.3.1) and (5.5.1)). Theorem 5.5.7 sums up all the information we have obtained on join irreducibles. Minimum condition for $\Lambda(\mathfrak{A} \circ \mathfrak{A})$ is derived in theorem 5.4.10.

Much of this chapter is closely modelled on the work of L. G. Kovács & M. F. Newman on varieties of metabelian groups. In particular I would mention (5.1.2) and its consequences, notably (5.2.3), and the fact, exploited several times, that the torsion subgroup of $A_1(\boldsymbol{F}_\infty(\mathscr{B}))$ (where $\mathscr{B} \leqslant \mathfrak{A} \circ \mathfrak{A}$) must have finite exponent. Statements of the Kovács & Newman results are included as (6.1.1) and (6.1.2); the close analogy with (5.2.3) and (5.5.1) is obvious. I want to stress that, though some of the details here are different from theirs, the underlying philosophy of this chapter is that of Kovács & Newman.

5.1. *Preliminaries*

A number of lemmas, necessary to the later parts of this chapter, will be proved here.

Lemma. *If m, t are coprime, then the bigroup $C_m \operatorname{wr} C_t$ generates $\mathfrak{A}_m \circ \mathfrak{A}_t$. The bigroup $C_m \operatorname{wr} C$ generates $\mathfrak{A}_m \circ \mathfrak{A}$. (Here C_m, C_t are cycles of order m, t, and C is an infinite cycle.)* (5.1.1)

Proof. Let $\boldsymbol{G}$ be critical in $\mathfrak{A}_m \circ \mathfrak{A}_t$; then if either $A_1(\boldsymbol{G})$ or $A_2(\boldsymbol{G}) = 1$, $\boldsymbol{G}$ is in svar$\{C_m \operatorname{wr} C_t\}$. If $A_1(\boldsymbol{G})$, $A_2(\boldsymbol{G})$ are non-trivial then by (3.2.1), $A_2(\boldsymbol{G})$ is cyclic, and $A_1(\boldsymbol{G})$ is generated *qua* $A_2(\boldsymbol{G})$-group by a single element; hence since $C_m \operatorname{wr} C_t$ is the split-free bigroup of rank (1, 1) in $\mathfrak{A}_m \circ \mathfrak{A}_t$, $\boldsymbol{G}$ is an epimorphic image of $C_m \operatorname{wr} C_t$. That is, $\mathfrak{A}_m \circ \mathfrak{A}_t$ is generated by $C_m \operatorname{wr} C_t$.

To prove the rest, suppose that $\{t_1, t_2, \ldots\}$ is an infinite set of natural numbers all prime to m, with $t_i | t_{i+1}$ for all $i \in I^+$. We show that $\mathfrak{A}_m \circ \mathfrak{A} = \bigvee\{\mathfrak{A}_m \circ \mathfrak{A}_{t_i} : i = 1, 2, \ldots\}$; clearly this implies that $C_m \operatorname{wr} C$ generates $\mathfrak{A}_m \circ \mathfrak{A}$. Consider the descending chain

$$A_2(\boldsymbol{W})^{t_1}[A_1(\boldsymbol{W}), A_2(\boldsymbol{W})^{t_1}] \geqslant A_2(\boldsymbol{W})^{t_2}[A_1(\boldsymbol{W}), A_2(\boldsymbol{W})^{t_2}] \geqslant \ldots$$

of biverbal sub-bigroups of $\boldsymbol{W} = \boldsymbol{F}_\infty(\mathfrak{A}_m \circ \mathfrak{A})$; these biverbal sub-bigroups are those corresponding to the bivarieties $\mathfrak{A}_m \circ \mathfrak{A}_{t_i}$. Now the chain

$$A_2(\boldsymbol{W})^{t_1} \geqslant A_2(\boldsymbol{W})^{t_2} \geqslant \ldots$$

has trivial intersection, and therefore, since the support of an element of $[A_1(\boldsymbol{W}), A_2(\boldsymbol{W})^{t_i}]$ is contained in $A_2(\boldsymbol{W})^{t_i}$, the chain

$$[A_1(\boldsymbol{W}), A_2(\boldsymbol{W})^{t_1}] \geqslant [A_1(\boldsymbol{W}), A_2(\boldsymbol{W})^{t_2}] \geqslant \ldots$$

also has trivial intersection. This concludes the proof.

The next lemma is a trivial adaptation of an unpublished result of L. G. Kovács about varieties of metabelian groups.

Lemma. *If $\mathscr{U}$ is a subvariety of $\mathfrak{A} \circ \mathfrak{A}$ which does not contain $\mathfrak{A}_m \circ \mathfrak{A}$, then all bigroups in $\mathscr{U}$ satisfy the bilaw*

$$[y_1, rz_1^s]^t,$$

for some integers r, s, t with $m \nmid t$. (5.1.2)

Proof. Consider the split-free bigroup of rank ∞ in $\mathscr{U}$, call it $\boldsymbol{W}$ say, on the free generating set $\{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\}$. Then

$$\prod_{i=0}^{v} a_1^{\alpha_i b_1^i} = 1 \tag{5.1.3}$$

for some integers $\alpha_0, \ldots, \alpha_v$ with α_0 not divisible by m; for, if there is no such relation holding, the sub-bigroup of $\boldsymbol{W}$ carried by $\langle a_1, b_1 \rangle$ is isomorphic to $C_m \operatorname{wr} C$ which generates $\mathfrak{A}_m \circ \mathfrak{A}$, by (5.1.1).

From (5.1.3) we deduce that

$$\prod_{i=0}^{v} a_1^{\alpha_i b_1^{ij}} = 1, \quad j \in \{0, \ldots, v\}.$$

Working in the endomorphism ring of $A_1(\boldsymbol{W})$ we have

$$\sum_{i=0}^{v} \alpha_i b_1^{ij} = 0, \quad j \in \{0, \ldots, v\}.$$

This implies $\alpha_0 \Pi_{j<i}(b_1^i - b_1^j) = 0$ and so $\alpha_0 \Pi_{j<i}(b_1^{i-j} - 1) = 0$. Hence

$$\alpha_0 \prod_{k=1}^{v} (b_1^k - 1)^{v-k+1} = 0,$$

whence

$$\alpha_0 (b_1^{v!} - 1)^{\frac{1}{2}v(v+1)} = 0.$$

Put $r = \frac{1}{2}v(v+1)$, $s = v!$, $t = \alpha_0$ and we have

$$[a_1, r b_1^s]^t = 1.$$

Lemma. *If $\mathscr{U}$ is a proper subvariety of $\mathfrak{A} \circ \mathfrak{A}$ then there exist natural numbers c, s, u such that $\mathscr{U}$ has a bilaw $[y_1, z_1^s, \ldots, z_c^s]^u$.* (5.1.4)

Proof. The argument is similar to one of Gupta & Newman (1966). By (5.1.2) $\mathscr{U}$ has a bilaw $[y_1, r z_1^s]^t$. Hence (assuming $r \geqslant 2$ and using (0.2.2) modulo $\boldsymbol{U}$)

$$[y_1, r z_1^s z_2^s, (r-2) z_2^s]^t = [y_1, (r-1) z_1^s, (r-1) z_2^s]^{rt}$$

is a bilaw in $\mathscr{U}$. Substituting a product for each of $z_1, z_2, \ldots$ in turn we deduce finally that

$$[y_1, z_1^s, \ldots, z_c^s]^u$$

is a bilaw in $\mathscr{U}$ where $c = 2^{r-1}$ and $u = t.r(r-1)^2(r-2)^4 \ldots 2^{r-2}$.

Lemma. *Let $\boldsymbol{F} = \boldsymbol{F}_\infty(\mathfrak{A} \circ \mathfrak{A})$ be freely generated by $\{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\}$. The commutators*

$$[a_i, j_1 b_1, \ldots, j_v b_v], \quad j_1, \ldots, j_v \geqslant 0, \quad \textit{and} \quad v > 0 \textit{ implies } j_v > 0,$$

$$j_1 + \ldots + j_v \leqslant c - 1 \quad (c \geqslant 1) \tag{5.1.5}$$

are linearly independent modulo $F_{(c+1)}$ and generate $A_1(\boldsymbol{F})$ modulo $F_{(c+1)}$. (5.1.6)

Corollary. *$\boldsymbol{F}/F_{(c+1)} \cong \boldsymbol{F}_\infty(\mathscr{N}_c \wedge \mathfrak{A} \circ \mathfrak{A})$ is torsion-free.* (5.1.7)

Corollary. *$F_{(c+1)}$ is complemented in $A_1(\boldsymbol{F})$.* (5.1.8)

Proof of (5.1.6). We use induction on c to prove that these commutators generate $A_1(\boldsymbol{F})$ modulo $F_{(c+1)}$. For b in $A_2(\boldsymbol{F})$,

$$a_i^b = a_i[a_i, b]$$

and therefore $\{a_1, a_2, \ldots\}$ generate $A_1(\boldsymbol{F})$ modulo $F_{(2)}$. Suppose therefore that the assertion is proved for $c-1$, and let q be an element of $F_{(c)} - F_{(c+1)}$. Then q can be written, modulo $F_{(c+1)}$, as a product of powers of commutators of the type

$$[a_i, j_1 b_1^{\pm 1}, \ldots, j_v b_v^{\pm 1}],$$

where $j_1+\ldots+j_v = c-1$. Using the identity $[x, y^{-1}] = ([x, y][x, y, y^{-1}])^{-1}$ we may then write q, modulo $F_{(c+1)}$, as a product of powers of commutators of the type

$$[a_i, j_1 b_1, \ldots, j_v b_v],$$

where $j_1+\ldots+j_v = c-1$. Combining this fact with the inductive hypothesis completes the proof except for the independence.

Suppose that we have a linear relation among the commutators (5.1.5) modulo $F_{(c+1)}$ which, by using appropriate deletions and renaming variables as necessary, we may assume to be of the form

$$\prod_{k=1}^{u} [a_1, j_{k1} b_1, \ldots, j_{kv} b_v]^{\alpha_k} \in F_{(c+1)},$$

where $0 < j_{kl}$ for all k, l and $\sum_{l=1}^{v} j_{kl} \leqslant c-1$. Let p be a prime greater than

$$\max\{j_{kl}, \alpha_k : 1 \leqslant k \leqslant u,\ 1 \leqslant l \leqslant v\}$$

and consider the natural homomorphism of $\boldsymbol{F}$ onto $\boldsymbol{F}_\infty(\mathfrak{A}_p \circ \mathfrak{A}_p)$: (4.3.5) then yields that α_k are all zero as required.

THEOREM. *Let s, c be natural numbers. Then the bivariety* $(\mathcal{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A}) \vee \mathfrak{A}_t \circ \mathfrak{A}$ *is determined modulo the bilaws of* $\mathfrak{A} \circ \mathfrak{A}$ *by the bilaw* $[y_1, z_1^s, \ldots, z_c^s]^t$. (5.1.9)

Proof. $\mathcal{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A}$ is certainly determined modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$ by $[y_1, z_1^s, \ldots, z_c^s]$. Now if $\boldsymbol{F} = \boldsymbol{F}_\infty(\mathfrak{A} \circ \mathfrak{A})$,

$$A_1(\mathfrak{E} \circ \mathfrak{A}_s(\boldsymbol{F})) = A_1(\boldsymbol{F})$$

and by Neumann (1964, lemma 8.2)

$$\mathfrak{E} \circ \mathfrak{A}_s(\boldsymbol{F}) \cong \boldsymbol{F}.$$

Hence by (5.1.8) $(\mathcal{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A})(\boldsymbol{F}) = \mathcal{N}_c(\mathfrak{E} \circ \mathfrak{A}_s(\boldsymbol{F}))$ is complemented in $A_1(\boldsymbol{F})$. Therefore

$$A_1(\boldsymbol{F})^t \cap (\mathcal{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A})(\boldsymbol{F}) = \{(\mathcal{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A})(\boldsymbol{F})\}^t$$

as required.

LEMMA. *Let Π be an infinite set of primes and* $\boldsymbol{H}$ *a nilpotent, finitely generated torsion-free bigroup. Then* $\boldsymbol{H}$ *is residually of prime* $(\in \Pi)$ *exponent.* (5.1.10)

Proof. The corresponding result for *groups* is true (cf. Higman 1955). Hence if h is a non-trivial element of H there is a normal subgroup N_h of H avoiding h such that H/N_h has exponent in Π, and $\cap\{N_h : 1 \neq h \in H\} = 1$. Write

$$M_h = ((A_1(\boldsymbol{H}) \cap N_h)(A_2(\boldsymbol{H}) \cap N_h))^H,$$

so that M_h carries a normal sub-bigroup of $\boldsymbol{H}$. Now for $i = 1, 2$,

$$A_i(\boldsymbol{H}) \cap M_h \leqslant A_i(\boldsymbol{H}) \cap N_h \leqslant A_i(\boldsymbol{H}) \cap M_h$$

so that $A_i(\boldsymbol{H}) \cap N_h = A_i(\boldsymbol{M}_h)$. Hence if H/N_h has exponent p, H/M_h is a p-group; and since we may assume p greater than the class of H, H/M_h is regular. Therefore, if a, b are elements of $A_1(\boldsymbol{H})$, $A_2(\boldsymbol{H})$ respectively

$$(ba)^p = b^p a^p c^p \pmod{M_h}$$

where c is in $A_1(\boldsymbol{H})$, and so $(ba)^p$ belongs to M_h. That is H/M_h has exponent p. Clearly $\cap\{M_h : 1 \neq h \in H\} = 1$.

5.2. *Torsion-free subvarieties of* $\mathfrak{A} \circ \mathfrak{A}$

Following L. G. Kovács & M. F. Newman we make the following definition.

Definition. *A bivariety is* torsion-free *if it is generated by bigroups* $\mathbf{G}$ *for all of which* $A_1(\mathbf{G})$ *is torsion-free.* (5.2.1)

One easily has then

Theorem. *A bivariety* $\mathscr{B}$ *is torsion-free if and only if* $A_1(\mathbf{F})$ *is torsion-free for every split-free bigroup* $\mathbf{F}$ *of* $\mathscr{B}$. (5.2.2)

In this section we begin a classification of all torsion-free subvarieties of $\mathfrak{A} \circ \mathfrak{A}$, a necessary step in our attempt at a classification of all subvarieties; it will be continued in § 5.3. The main theorem is the following one.

Theorem. *The bivarieties* $\mathfrak{E} \circ \mathfrak{A}$, $\mathfrak{E} \circ \mathfrak{A}_s$, $\mathfrak{A} \circ \mathfrak{A}_s$, $\mathscr{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A}$ $(s \geq 1, c \geq 2)$ *are all torsion-free, and every proper torsion-free subvariety of* $\mathfrak{A} \circ \mathfrak{A}$ *can be written as a finite join of these special ones.* (5.2.3)

Proof. It is obvious that $\mathfrak{E} \circ \mathfrak{A}_s$, $\mathfrak{A} \circ \mathfrak{A}_s$ are torsion-free; and for $\mathscr{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A}$ $(c \geq 2)$ note that, if $\mathbf{F} = \mathbf{F}_\infty(\mathfrak{A} \circ \mathfrak{A})$, then since

$$\mathfrak{E} \circ \mathfrak{A}_s(\mathbf{F}) \cong \mathbf{F}$$

(by Neumann (1964, lemma 8.1)),

$$A_1(\mathbf{F}_\infty(\mathscr{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A})) \cong A_1(\mathbf{F}_\infty(\mathscr{N}_c \wedge \mathfrak{A} \circ \mathfrak{A})),$$

which is torsion-free (5.1.7).

Let $\mathscr{B}$ be a proper torsion-free subvariety of $\mathfrak{A} \circ \mathfrak{A}$. It follows from (5.1.4) that $\mathscr{B}$ has a bilaw $[y_1, z_1^s, \ldots, z_c^s]$ for some natural numbers s, c; in the case when $\mathscr{B}$ has a bilaw z_1^n choose s to be the least such n. Write $\mathbf{F}_r = \mathbf{F}_{(r,r)}(\mathscr{B})$ $(r \geq 1)$ and then

$$\mathbf{H}_r = \mathfrak{E} \circ \mathfrak{A}_s(\mathbf{F}_r)$$

is finitely generated, torsion-free and nilpotent. Let Π be an infinite set of primes (say all chosen greater than c) so that from (5.1.10) there exist normal sub-bigroups $\mathbf{N}_r(p)$ of $\mathbf{H}_r$, for each p in Π, such that

$$\exp(\mathbf{H}_r/\mathbf{N}_r(p)) = p, \quad \cap \{\mathbf{N}_r(p) : p \in \Pi\} = 1.$$

Moreover, if we choose $\mathbf{N}_r(p)$ to be the smallest such sub-bigroup of $\mathbf{H}_r$ then $\mathbf{N}_r(p)$ is normal in $\mathbf{F}_r$; and if Π' is an infinite subset of Π then $\cap \{\mathbf{N}_r(p) : p \in \Pi'\}$ is still equal to 1. We have

$$\operatorname{svar} \mathbf{F}_r = \bigvee \{\operatorname{svar} \mathbf{F}_r/\mathbf{N}_r(p) : p \in \Pi\}. \tag{5.2.4}$$

We can now employ some results from the last chapter. For,

$$\mathfrak{E} \circ \mathfrak{A}_s \leq \operatorname{svar} \mathbf{F}_r/\mathbf{N}_r(p) \leq \mathfrak{A}_p \circ \mathfrak{A}_{ps}, \quad p \in \Pi$$

and by (4.1.8) there exists to each divisor t of s a subvariety $\mathscr{S}_t(r,p)$ of $\mathfrak{A}_p \circ \mathfrak{A}_p$ such that

$$\operatorname{svar} \mathbf{F}_r/\mathbf{N}_r(p) = \bigvee \{\mathscr{S}_t(r,p)\ (\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A} : t|s\}. \tag{5.2.5}$$

Now by (4.4.4) we can determine $\mathscr{S}_t(r,p)$: for $t > 1$,

$$\mathscr{S}_t(r,p) \in \{\mathfrak{E} \circ \mathfrak{E}, \mathfrak{A}_p \circ \mathfrak{E}, \mathfrak{A}_p \circ \mathfrak{A}_p \wedge \mathscr{N}_{d(t,r,p)}\},$$

and $\quad \mathscr{S}_1(r,p) \in \{\mathfrak{E} \circ \mathfrak{E}, \mathfrak{A}_p \circ \mathfrak{E}, \mathfrak{E} \circ \mathfrak{A}_p, \mathfrak{A}_p \circ \mathfrak{A}_p \wedge \mathscr{N}_{d(1,r,p)}\},$

where $c \geq d(t,r,p) \geq 1$ and for $t > 1$, $d(t,r,p) \geq 2$. Since the number of choices of $\mathscr{S}_t(r,p)$ is uniformly bounded for all p, there exists an infinite subset Π' of Π such that for all p in Π' and each t dividing s, either

$$\mathscr{S}_t(r,p) = \mathfrak{E}\circ\mathfrak{E} \quad\text{or}\quad \mathscr{S}_t(r,p) = \mathfrak{A}_p\circ\mathfrak{E} \quad\text{or}\quad \mathscr{S}_t(r,p) = \mathfrak{E}\circ\mathfrak{A}_p$$

or

$$\mathscr{S}_t(r,p) = \mathfrak{A}_p\circ\mathfrak{A}_p\wedge\mathscr{N}_{d(r,t)}$$

(where $d(r,t) \leqslant c$). Combining (5.2.4) and (5.2.5) and rearranging terms

$$\operatorname{svar}\boldsymbol{F}_r = \bigvee\{\mathscr{S}_t(r,p)\ (\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : p\in\Pi', t|s\}. \tag{5.2.6}$$

Now

$$\begin{aligned}\bigvee\{\mathscr{S}_t(r,p)\ (\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : p\in\Pi'\} \\ &= \mathfrak{E}\circ\mathfrak{A}_t \quad\text{if}\quad \mathscr{S}_t(r,p) = \mathfrak{E}\circ\mathfrak{E},\\ &= \mathfrak{E}\circ\mathfrak{A} \quad\text{if}\quad \mathscr{S}_t(r,p) = \mathfrak{E}\circ\mathfrak{A}_p,\\ &\leqslant \mathfrak{A}\circ\mathfrak{A}_t \quad\text{if}\quad \mathscr{S}_t(r,p) = \mathfrak{A}_p\circ\mathfrak{E}\\ &\leqslant \mathscr{N}_{d(r,t)}(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} \quad\text{if}\quad \mathscr{S}_t(r,p) = \mathfrak{A}_p\circ\mathfrak{A}_p\wedge\mathscr{N}_{d(r,t)};\end{aligned} \tag{5.2.7}$$

and, since $\mathfrak{A}\circ\mathfrak{A}_t$, $\mathscr{N}_{d(r,t)}(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A}$ are torsion-free, we may employ exactly the same technique to deduce that

$$\begin{aligned}\mathfrak{A}\circ\mathfrak{A}_t &\leqslant \bigvee\{(\mathfrak{A}_p\circ\mathfrak{E})\ (\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : p\in\Pi'\},\\ \mathscr{N}_{d(r,t)}(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} &\leqslant \bigvee\{(\mathfrak{A}_p\circ\mathfrak{A}_p\wedge\mathscr{N}_{d(r,t)})\ (\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : p\in\Pi'\},\end{aligned} \tag{5.2.8}$$

whence

$$\begin{aligned}\bigvee\{\mathscr{S}_t(r,p)\ (\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : p\in\Pi'\} \\ \in\{\mathfrak{E}\circ\mathfrak{A}_t, \mathfrak{A}\circ\mathfrak{A}_t, \mathfrak{E}\circ\mathfrak{A}, \mathscr{N}_{d(r,t)}(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A}\}.\end{aligned}$$

Substituting in (5.2.6) we see that $\operatorname{svar}\boldsymbol{F}_r$ is a finite join of the kind asserted in the theorem; note that the number of choices for constituents in this join is uniformly bounded for all r. Hence, since

$$\mathscr{B} = \bigvee\{\operatorname{svar}\boldsymbol{F}_r : r\geqslant 1\},$$

we may express $\mathscr{B}$ as a finite join of elements from the set

$$\{\mathfrak{E}\circ\mathfrak{A}_t, \mathfrak{E}\circ\mathfrak{A}, \mathfrak{A}\circ\mathfrak{A}_t, \mathscr{N}_d(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : t|s,\ 2\leqslant d\leqslant c\},$$

thus proving (5.2.3).

Corollary. *If Π is an infinite set of primes then*

$$\begin{aligned}\mathfrak{A}\circ\mathfrak{A}_t &= \bigvee\{\mathfrak{A}_p\circ\mathfrak{A}_t : p\in\Pi\},\\ \mathscr{N}_d(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} &= \bigvee\{(\mathfrak{A}_p\circ\mathfrak{A}_p\wedge\mathscr{N}_d)\ (\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A} : p\in\Pi\}.\end{aligned} \tag{5.2.9}$$

Proof. This follows from (5.2.7) and (5.2.8).

5.3. *Subvarieties of* $\mathfrak{A}\circ\mathfrak{A}_n$ *and* $\mathfrak{A}_m\circ\mathfrak{A}$

The main results in this section will be the following theorem and (5.3.8).

Theorem. *If n is a natural number and $\mathscr{B}$ is a subvariety of $\mathfrak{A}\circ\mathfrak{A}_n$, then there exists a subset Δ of the divisors of n, and a bivariety $\mathscr{P}$ of finite exponent such that*

$$\mathscr{B} = \bigvee\{\mathfrak{A}\circ\mathfrak{A}_\delta : \delta\in\Delta\}\vee\mathscr{P}.$$

Moreover if Δ is chosen to be minimal then it is unique. (5.3.1)

COROLLARY. *If n is a natural number, then* $\Lambda(\mathfrak{A}\circ\mathfrak{A}_n)$ *has minimum condition.*

Proof. If $n > 1$, and $\mathscr{B}_0$ is a proper subvariety of $\mathfrak{A}\circ\mathfrak{A}_n$, then by (5.3.1)

$$\mathscr{B}_0 = \bigvee\{\mathfrak{A}\circ\mathfrak{A}_\delta : \delta\in\Delta\}\vee\mathscr{P};$$

and if $\mathscr{B}_0 \geqslant \mathscr{B}_1 \geqslant \ldots \geqslant \mathscr{B}_i \geqslant \ldots$ is a descending chain we may assume that the unique subset of the divisors of n described in (5.3.1) is the same for all i. Hence, using the modular law in $\Lambda(\mathfrak{A}\circ\mathfrak{A}_n)$

$$\mathscr{B}_i = \bigvee\{\mathfrak{A}\circ\mathfrak{A}_\delta : \delta\in\Delta\}\vee(\mathscr{P}\wedge\mathscr{B}_i), \tag{5.3.2}$$

and by (4.2.29) the chain $\mathscr{P}\wedge\mathscr{B}_0 \geqslant \mathscr{P}\wedge\mathscr{B}_1 \geqslant \ldots$ must break off. Therefore the chain $\mathscr{B}_0 \geqslant \mathscr{B}_1 \geqslant \ldots$ must break off also.

To prove (5.3.1) we need three lemmas and some results from chapter 4.

LEMMA. *Let* Δ *be a finite set of natural numbers. Then* $\mathscr{B}(\Delta) = \bigvee\{\mathfrak{A}\circ\mathfrak{A}_\delta : \delta\in\Delta\}$ *is finitely based.* (5.3.3)

Proof. With each subset Γ of Δ associate its least common multiple l_Γ and the biword

$$q_\Gamma = [y_1, z_1^{l_\Gamma}, z_2^{\delta_1}, \ldots, z_{r+1}^{\delta_r}],$$

where $\{\delta_1, \ldots, \delta_r\} = \Delta - \Gamma$. We shall prove by induction on $|\Delta|$ that $\mathscr{B}(\Delta)$ has

$$S = \{q_\Gamma : \Gamma\subseteq\Delta\}\cup\{z_1^{l_\Delta}\}$$

as a basis modulo the bilaws of $\mathfrak{A}\circ\mathfrak{A}$.

For $|\Delta| = 1$ the result is obvious. Suppose therefore that $|\Delta| > 1$ and that the result is proved for all proper subsets of Δ. Let $\mathscr{V}(\Delta)$ be the sub-bivariety of $\mathfrak{A}\circ\mathfrak{A}$ determined by S, and let $\mathbf{G}$ in $\mathscr{V}(\Delta)$ be critical. If G is nilpotent it is a p-group and $\mathbf{G}$ has a bilaw $z_1^{p^\alpha}$, where $p^\alpha | l_\Delta$, and since this implies that $p^\alpha|\delta$ for some $\delta\in\Delta$, $\mathbf{G}\in\mathfrak{A}\circ\mathfrak{A}_\delta\leqslant\mathscr{B}(\Delta)$. If G is not nilpotent write $\Gamma = \{\delta\in\Delta : |K|\,|\delta\}$. Then if $\Gamma \neq \Delta$ it follows from the fact that K acts fixed-point-free on $A_1(\mathbf{G})$, that the biwords

$$z_1^{l_\Gamma},\quad q'(\Xi) = [y_1, z_1^{l_\Xi}, z_2^{\gamma_1}, \ldots, z_s^{\gamma_s}],\quad \Xi\subseteq\Gamma,\quad \{\gamma_1, \ldots, \gamma_s\} = \Gamma-\Xi,$$

are all bilaws in $\mathbf{G}$. Hence $\mathbf{G}$ belongs to $\mathscr{V}(\Gamma)$ and so by induction, $\mathbf{G}\in\mathscr{B}(\Gamma)\leqslant\mathscr{B}(\Delta)$; finally if $\Gamma = \Delta$, then since $\exp A_2(\mathbf{F}) = p^\beta$ (where p is prime not dividing $|K|$) it follows that $p^\beta | l_\Delta$, as before that $p^\beta|\delta$ for some δ in Δ, and therefore since $|K|\,|\delta$, that $\exp A_2(\mathbf{G})|\delta$. This completes the proof.

The foregoing proof yields

COROLLARY. *If* $\mathbf{G}\in\bigvee\{\mathfrak{A}\circ\mathfrak{A}_\delta : \delta\in\Delta\}$ *is critical then* $\mathbf{G}$ *belongs to* $\mathfrak{A}\circ\mathfrak{A}_\delta$ *for some* δ *in* Δ. (5.3.4)

LEMMA. *The torsion-free subvarieties of* $\mathfrak{A}\circ\mathfrak{A}_n$ *are finite joins of elements of the set*

$$\{\mathfrak{E}\circ\mathfrak{A}_\delta, \mathfrak{A}\circ\mathfrak{A}_\delta : \delta|n\}. \tag{5.3.5}$$

Proof. This follows at once from (5.2.3) since for $c \geqslant 2$

$$\mathscr{N}_c(\mathfrak{E}\circ\mathfrak{A}_\delta)\wedge\mathfrak{A}\circ\mathfrak{A} \nleqslant \mathfrak{A}\circ\mathfrak{A}_n.$$

Proof of (5.3.1). Suppose, without loss of generality, that $\mathfrak{E}\circ\mathfrak{A}_n \leqslant \mathscr{B} \leqslant \mathfrak{A}\circ\mathfrak{A}_n$ and that $\mathscr{B}$ does not have finite exponent. Write $\mathbf{F} = \mathbf{F}_\infty(\mathscr{B})$ and let T be the torsion subgroup of $A_1(\mathbf{F})$. Then svar $\mathbf{F}/\mathbf{T}$ is torsion-free and therefore, by (5.3.5), for some Δ

$$\text{svar}\,\mathbf{F}/\mathbf{T} = \bigvee\{\mathfrak{A}\circ\mathfrak{A}_\delta : \delta\in\Delta\}\vee\mathfrak{E}\circ\mathfrak{A}_n, \tag{5.3.6}$$

which, with the use of (5.3.3), is easily checked to be finitely based. Hence T is the closure of finitely many elements and therefore has finite exponent, m say. Finally note that

$$A_1(\boldsymbol{F})^m \cap T = 1$$

and therefore

$$\mathscr{B} = \operatorname{svar} \boldsymbol{F}/\boldsymbol{T} \vee \mathscr{P}', \tag{5.3.7}$$

where $\mathscr{P}'(\leqslant \mathfrak{A}_m \circ \mathfrak{A}_n)$ has finite exponent. Put $\mathscr{P} = \mathscr{P}' \vee \mathfrak{E} \circ \mathfrak{A}_n$ and combine (5.3.5) and (5.3.7) and we have finished except for the uniqueness of a minimal Δ.

Suppose Δ is not empty, that $\delta_1, \delta_2 \in \Delta$, $\delta_1 | \delta_2$ implies $\delta_1 = \delta_2$ and that

$$\mathfrak{A} \circ \mathfrak{A}_\zeta \leqslant \bigvee \{\mathfrak{A} \circ \mathfrak{A}_\delta : \delta \in \Delta\} \vee \mathscr{P},$$

where $\mathscr{P}$ has finite exponent, N say. Let q be a commutator bilaw of $\bigvee\{\mathfrak{A} \circ \mathfrak{A}_\delta : \delta \in \Delta\}$ so that q^N is a bilaw of the join and therefore of $\mathfrak{A} \circ \mathfrak{A}_\zeta$. Since $\mathfrak{A} \circ \mathfrak{A}_\zeta$ is torsion-free, q is a bilaw of $\mathfrak{A} \circ \mathfrak{A}_\zeta$. Hence

$$\mathfrak{A} \circ \mathfrak{A}_\zeta \leqslant \bigvee \{\mathfrak{A} \circ \mathfrak{A}_\delta : \delta \in \Delta\}$$

(if z_1^γ is a bilaw of $\bigvee\{\mathfrak{A} \circ \mathfrak{A}_\delta : \delta \in \Delta\}$ consider $q = [y_1, z_1^\gamma]$). Now if $\boldsymbol{G}$ in $\mathfrak{A} \circ \mathfrak{A}_\zeta$ is non-nilpotent and critical with $|K| = \zeta$ then, by (5.3.4), $\boldsymbol{G}$ belongs to $\mathfrak{A} \circ \mathfrak{A}_\delta$ for some $\delta \in \Delta$. This shows that $\zeta | \delta$ and therefore that $\mathfrak{A} \circ \mathfrak{A}_\zeta \leqslant \mathfrak{A} \circ \mathfrak{A}_\delta$. Thus if

$$\bigvee \{\mathfrak{A} \circ \mathfrak{A}_\delta : \delta \in \Delta\} \vee \mathscr{P} = \bigvee \{\mathfrak{A} \circ \mathfrak{A}_\gamma : \gamma \in \Gamma\} \vee \mathscr{P}',$$

where $\mathscr{P}$, $\mathscr{P}'$ have finite exponent, then if $\delta \in \Delta$ there exists $\gamma \in \Gamma$ such that $\delta | \gamma$, and there exists $\delta' \in \Delta$ such that $\gamma | \delta'$, whence $\delta | \delta'$ and so $\delta = \delta' = \gamma$; that is $\Delta \leqslant \Gamma$. Similarly, $\Gamma \leqslant \Delta$ and we are finished.

Finally, in this section, we shall prove

THEOREM. *If m is a natural number and $\mathscr{B}$ a subvariety of $\mathfrak{A}_m \circ \mathfrak{A}$ then there exists a unique $u | m$, and a bivariety $\mathscr{P}$ of finite exponent such that*

$$\mathscr{B} = \mathfrak{A}_u \circ \mathfrak{A} \vee \mathscr{P}. \tag{5.3.8}$$

COROLLARY. *$\Lambda(\mathfrak{A}_m \circ \mathfrak{A})$ has minimum condition.*

Proof. The proof is similar to that of (5.3.2) and we omit it. (5.3.9)

Proof of (5.3.8). Let $\mathscr{B}$ be a proper subvariety of $\mathfrak{A}_m \circ \mathfrak{A}$. Then according to (5.1.2) there exists a smallest natural number u ($m \nmid u$) such that for some r, s $\mathscr{B}$ has a bilaw $[y_1, rz_1^s]^u$. Then $\mathfrak{A}_u \circ \mathfrak{A} \leqslant \mathscr{B}$; for if not, (5.1.2) would lead to a law $[y_1, r'z_1^{s'}]^t$ in B with $t < u$. Now $\mathscr{B}$ has a bilaw $[y_1, z_1^{sm^r}]^u$ since, using (0.2.1), we have

$$[y_1, z_1^{sm^r}]^u = \prod_{i=1}^{m^r} [y_1, iz_1^s]^{u\binom{m^r}{i}}$$
$$= \prod_{i=1}^{r-1} [y_1, iz_1^s]^{u\binom{m^r}{i}},$$

and for $1 \leqslant i \leqslant r-1$, $m | \binom{m^r}{i}$. Hence by (5.1.9)

$$\mathfrak{A}_u \circ \mathfrak{A} \leqslant \mathscr{B} \leqslant \mathfrak{A}_u \circ \mathfrak{A} \vee \mathfrak{A} \circ \mathfrak{A}_{sm^r},$$

and modularity then gives

$$\mathscr{B} = \mathfrak{A}_u \circ \mathfrak{A} \vee (\mathscr{B} \wedge \mathfrak{A} \circ \mathfrak{A}_{sm^r}).$$

Finally note that $\mathscr{B} \wedge \mathfrak{A} \circ \mathfrak{A}_{sm^r} \leqslant \mathfrak{A}_m \circ \mathfrak{A} \wedge \mathfrak{A} \circ \mathfrak{A}_{sm^r} = \mathfrak{A}_m \circ \mathfrak{A}_{sm^r}$. This completes the proof.

5.4. *Minimum condition for* $\Lambda(\mathfrak{A} \circ \mathfrak{A})$

Let $\boldsymbol{F} = \boldsymbol{F}_\infty(\mathfrak{A} \circ \mathfrak{A})$ be freely generated by $\{y_1, y_2, \ldots\} \cup \{z_1, z_2, \ldots\}$ and let

$$\boldsymbol{V}(c,s) = \mathcal{N}_c(\mathfrak{C} \circ \mathfrak{A}_s)(\boldsymbol{F}) \; (c, s \geqslant 1).$$

Then we have the following rather technical lemma.

Lemma. *Let* a_i *in* $V(c-i, s) \cap F_{(\infty,\nu)}(\mathfrak{A} \circ \mathfrak{A})$ $(0 \leqslant i \leqslant c)$ *be such that, if* $\alpha_k : \boldsymbol{F} \to \boldsymbol{F}$ *is defined for* $k \in \{1, \ldots, c\}$ *by*

$$y_i \alpha_k = y_i, \quad z_1 \alpha_k = z_1 z_{\nu+k}^s, \quad z_j \alpha_k = z_j, \quad j \neq 1, \quad i, j \in I^+,$$

then, for all $b_1, \ldots, b_i \in A_2(\boldsymbol{F})^s$ *and* $k \in \{1, \ldots, c\}$,

$$[a_i, b_1, \ldots, b_i]\, \alpha_k = [a_i, b_1 \alpha_k, \ldots, b_i \alpha_k] \bmod V(c+1, s).$$

Then there exists a natural number N_1, *depending on* c, s *only, such that for* $i \in \{0, \ldots, c\}$

$$[a_i, z_{\nu+1}^s, \ldots, z_{\nu+i}^s]^{N_1} \in cl\left\{\prod_{j=0}^{c} [a_i, jz_1^s]\right\} . \, \boldsymbol{V}(c+1, s). \tag{5.4.1}$$

Proof. Let $d \in \{0, \ldots, c\}$ and make the following definitions: for $j \in \{0, \ldots, d\}$ write $a_j(0) = a_j$ and for $i \in \{1, \ldots, d\}$ write

$$a_j(i) = \prod_{l=j+1}^{d-i+1} [a_l(i-1), (l-j)\, z_{\nu+i}^{s^2}]^{\binom{l}{j}}, \quad j \in \{0, \ldots, d-i\};$$

and for $i \in \{0, \ldots, d\}$,

$$q_i(d) = \prod_{j=0}^{d-i} [a_j(i), jz_1^s].$$

It is easy to verify that $a_j(i) \in V(c-j, s) \cap F_{(\infty,\, \nu+i)}(\mathfrak{A} \circ \mathfrak{A})$ and that, for arbitrary $b_1, \ldots, b_j$ in $A_2(\boldsymbol{F})^s$,

$$[a_j(i), b_1, \ldots, b_j]\, \alpha_k = [a_j(i), b_1 \alpha_k, \ldots, b_j \alpha_k] \bmod V(c+1, s)$$

for $k \in \{i+1, \ldots, d\}$.

We prove that for $d \geqslant 1$ and $i \in \{0, \ldots, d-1\}$

$$q_{i+1}(d) \in (cl q_i(d))\, \boldsymbol{V}(c+1, s).$$

For,

$$q_i(d)\, \alpha_{i+1} = \prod_{j=0}^{d-i} [a_j(i), jz_1^s\, z_{\nu+i+1}^{s^2}]$$

$$= \prod_{j=0}^{d-i} \prod_{k=0}^{j} [a_j(i), kz_1^s, (j-k)\, z_{\nu+i+1}^{s^2}]^{\binom{j}{k}} \bmod V(c+1, s),$$

whence

$$\prod_{j=1}^{d-i} \prod_{k=0}^{j-1} [a_j(i), kz_1^s, (j-k)\, z_{\nu+i+1}^{s^2}]^{\binom{j}{k}} \in (cl q_i(d))\, \boldsymbol{V}(c+1, s).$$

Collecting terms on the left of this expression we find

$$\prod_{k=0}^{d-i-1} \left[\prod_{j=k+1}^{d-i} [a_j(i), (j-k)\, z_{\nu+i+1}^{s^2}]^{\binom{j}{k}}, kz_1^s \right] \in (cl q_i(d))\, \boldsymbol{V}(c+1, s),$$

or, in other words,

$$q_{i+1}(d) = \prod_{k=0}^{d-i-1} [a_k(i+1), kz_1^s] \in (cl q_i(d))\, \boldsymbol{V}(c+1, s).$$

Now

$$\begin{aligned} q_d(d) &= a_0(d) \\ &= [a_1(d-1), z_{\nu+d}^{s^2}]^1 = \ldots = [a_i(d-i), z_{\nu+d-i+1}^{s^2}, \ldots, z_{\nu+d}^{s^2}]^{i!} \\ &= [a_d(0), z_{\nu+1}^{s^2}, \ldots, z_{\nu+d}^{s^2}]^{d!} = [a_d, z_{\nu+1}^s, \ldots, z_{\nu+d}^s]^{s^d d!} \quad \bmod V(c+1, s) \end{aligned}$$

and so we have

$$[a_d, z_{\nu+1}^s, \ldots, z_{\nu+d}^s]^{d!s^d} \in (cl q_0(d))\, \boldsymbol{V}(c+1, s). \tag{5.4.2}$$

Now for $d \in \{1, \ldots, c\}$ define

$$N_d = \prod_{j=d}^{c} (j!\,s^j).$$

Then we shall prove by (downward) induction on d, that

$$[a_d, z_{\nu+1}^s, \ldots, z_{\nu+d}^s]^{N_d} \in (cl q_0(c))\, V(c+1, s); \tag{5.4.3}$$

when $d = c$ we have proved this in (5.4.2), and if for $d < j \leqslant c$

$$[a_j, z_{\nu+1}^s, \ldots, z_{\nu+j}^s]^{N_j} \in (cl q_0(c))\, V(c+1, s)$$

has been proved, then

$$q_0(d)^{N_{d+1}} = q_0(c) \bmod V(c+1, s)$$

and from (5.4.2), raising each side to the power N_{d+1}, we have

$$\begin{aligned}[a_d, z_{\nu+1}^s, \ldots, z_{\nu+d}^s]^{N_d} &\in (cl q_0(d)^{N_{d+1}})\, V(c+1, s)\\ &\leqslant (cl q_0(c))\, V(c+1, s).\end{aligned}$$

Finally, noting that N_d divides N_1 for all d, we see that (5.4.3) is what we set out to prove.

Lemma. *If $U \leqslant A_1(\boldsymbol{F})$ carries a fully invariant sub-bigroup of $\boldsymbol{F}$, and*

$$V(c+1, s) \leqslant U \leqslant V(c, s)$$

then there exists $\boldsymbol{U}^*$ fully invariant in $\boldsymbol{F}$, and a natural number N depending only on c, s such that

$$V(1, s) \leqslant U^*,$$

and

$$[U^*, cA_2(\boldsymbol{F})^s]^N \leqslant U \leqslant [U^*, cA_2(\boldsymbol{F})^s]. \tag{5.4.4}$$

Proof. Our proof will be by induction on c; for $c = 0$ $\boldsymbol{U}^* = \boldsymbol{U}$, $N = 1$ will do, so suppose that $c > 1$ and that for $d \in \{0, \ldots, c-1\}$ the result has been proved. Let q be an element of U, so that (modulo $V(c+1, s)$) q can be written as a product of powers of commutators of the type

$$[y_i, z_{i1}^{\beta_1}, \ldots, z_{ir}^{\beta_r}, z_{j_1}^s, \ldots, z_{j_c}^s],$$

where s does not divide $\beta_1, \ldots, \beta_r$. By renaming variables if necessary and collecting terms according to the number of repetitions of z_i^s, say, we may suppose $\{j_1, \ldots, j_c\} \subseteq \{1, \ldots, c\}$ and write

$$q = \prod_{j=0}^{c} [a_j(q, i), jz_i^s] \tag{5.4.5}$$

(filling up with dummy $a_j(q, i)$'s if necessary). Now $a_0(q, i), \ldots, a_c(q, i)$ satisfy the hypotheses of lemma 5.4.1 and so in particular

$$\begin{aligned}&cl\{[a_j(q, 1), jA_2(\boldsymbol{F})^s]^{N_1} : 0 \leqslant j \leqslant c\}\\ &\qquad \leqslant (cl q)\, V(c+1, s) \leqslant cl\{[a_j(q, 1), jA_2(\boldsymbol{F})^s] : 0 \leqslant j \leqslant c\}\, V(c+1, s).\end{aligned} \tag{5.4.6}$$

Define
$$\alpha_j(q, 1) = a_j(q, 1), \quad j \in \{0, \ldots, c\}$$

and for $k \in \{2, \ldots, c\}$,
$$\alpha_j(q, k) = a_j(\alpha_0(q, k-1), k), \quad j \in \{0, \ldots, c\}.$$

Apply (5.4.1) to each $\alpha_0(q, k)$ in turn as (5.4.6) was applied to q and deduce

$$\begin{aligned}&cl\{[\alpha_j(q, k), jA_2(\boldsymbol{F})^s]^{N_1^c} : 1 \leqslant j \leqslant c,\ 0 \leqslant k \leqslant c\}\\ &\qquad \leqslant (cl q)\, \boldsymbol{V}(c+1, s)\\ &\qquad \leqslant cl\{[\alpha_j(q, k), jA_2(\boldsymbol{F})^s] : 1 \leqslant j \leqslant c,\ 0 \leqslant k \leqslant c\}\, \boldsymbol{V}(c+1, s).\end{aligned} \tag{5.4.7}$$

Notice that $\alpha_0(q, c) = 1$. Since $\alpha_j(q, k) \in V(c-j, s)$ for all $j \in \{1, \ldots, c\}$, if we write

$$\boldsymbol{U}_j = cl\{\alpha_j(q, k) : q \in \boldsymbol{U},\ 0 \leqslant k \leqslant c\}\, V(c-j+1, s)$$

we may apply the inductive hypothesis to deduce the existence of U_j^* containing $V(1,s)$ and an integer n_j depending only on c, s, j such that

$$[U_j^*, (c-j)\,A_2(F)^s]^{n_j} \leqslant U_j \leqslant [U_j^*, (c-j)\,A_2(F)^s] \quad j\in\{1,\dots,c\}. \tag{5.4.8}$$

Hence if $N = N_1^c n_1 n_2 \dots n_c$, and $U^* = \Pi\{U_j^* : 1 \leqslant j \leqslant c\}$ then

$$V(1,s) \leqslant U^*$$

and, from (5.4.7) and (5.4.8),

$$[U^*, cA_2(F)^s]^N \leqslant U \leqslant [U^*, cA_2(F)^s]$$

as required. Note that N depends on c, s alone.

COROLLARY. *If* $V(c+1,s) \leqslant U \leqslant U^\dagger \leqslant V(c,s)$ *then*

$$U^* \leqslant (U^\dagger)^*. \tag{5.4.9}$$

Proof. For, by definition, $U_j \leqslant U_j^\dagger$ for $j\in\{1,\dots,c\}$ and hence, by induction, $U_j^* \leqslant (U_j^\dagger)^*$ whence $U^* \leqslant (U^\dagger)^*$.

THEOREM. $\Lambda(\mathfrak{A}\circ\mathfrak{A})$ *has minimum condition.* (5.4.10)

Proof. According to (5.1.4) and (5.1.8) a proper subvariety of $\mathfrak{A}\circ\mathfrak{A}$ is contained in some $(\mathscr{N}_c(\mathfrak{E}\circ\mathfrak{A}_s)\wedge\mathfrak{A}\circ\mathfrak{A})\vee\mathfrak{A}_t\circ\mathfrak{A}$, so that, in order to prove minimum condition for $\Lambda(\mathfrak{A}\circ\mathfrak{A})$, it suffices to consider the lattice of subvarieties of $\mathscr{V}(c,s) = \mathscr{N}_c(\mathfrak{E}\circ\mathfrak{A}_s)\wedge\mathfrak{A}\circ\mathfrak{A}$ for all c, $s\in I^+$, by virtue of (5.3.9) and (2.1.3). Moreover, by (2.1.4), we need consider descending chains between $\mathscr{V}(d,s)$ and $\mathscr{V}(d-1,s)$ only, for $d\in\{2,\dots,c\}$. (The case

$$\Lambda(\mathscr{N}_1(\mathfrak{E}\circ\mathfrak{A}_s)\wedge\mathfrak{A}\circ\mathfrak{A}) = \Lambda(\mathfrak{A}\circ\mathfrak{A}_s\vee\mathfrak{E}\circ\mathfrak{A})$$

is covered by (5.3.2) and (2.1.3).)

Suppose accordingly that

$$\mathscr{V}(d,s) > \mathscr{U}_1 \geqslant \mathscr{U}_2 \geqslant \dots \geqslant \mathscr{U}_i \geqslant \dots \geqslant \mathscr{V}(d-1,s) \tag{5.4.11}$$

is a descending chain. Since by (5.2.3) the torsion-free bivarieties between $\mathscr{V}(d-1,s)$ and $\mathscr{V}(d,s)$ have descending chain condition we may assume that the largest torsion-free subvariety contained in each $\mathscr{U}_i$ is the same; call it $\mathscr{T}$. If $F = F_\infty(\mathfrak{A}\circ\mathfrak{A})$ then the torsion subgroup T_1/U_1 of $A_1(F/U_1)$ is $\mathscr{T}(F/U_1)$. We show that T_1/U_1 has finite exponent. For, there exists an ascending chain $n_1|n_2|\dots|n_i|\dots$ in the division ordering of I^+ such that, if

$$W_i/U_1 = \{x\in T_1/U_1 : x^{n_i} = 1\}, \quad i\in I^+,$$

Then $T_1/U_1 = \cup\{W_i/U_1 : i\in I^+\}$, and each W_i is fully invariant in F. By (5.4.4) there exists an ascending chain

$$V(1,s) \leqslant W_1^* \leqslant \dots \leqslant W_i^* \leqslant \dots, \tag{5.4.12}$$

and a natural number N depending on d, s alone such that

$$[W_i^*, dA_2(F)^s]^N \leqslant W_i \leqslant [W_i^*, dA_2(F)^s].$$

By (5.3.2) and (2.1.3) the chain (5.4.12) breaks off; hence there exists a natural number m with $W_m^* = W_i^*$ $(i \geqslant m)$. Hence $i \geqslant m$ implies

$$W_i^N \leqslant [W_m^*, dA_2(F)^s]^N \leqslant W_m,$$

whence $(W_i/U_1)^N \leqslant W_m/U_1$; and therefore, if $x_i\in W_i/U_1$ has order n_i exactly, $x_i^{Nn_m} = 1$ and so

$$n_i \mid Nn_m, \quad i \geqslant m.$$

This proves that the exponent, e say, of T_1/U_1 divides Nn_m.

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We have then that

$$A_1(\boldsymbol{F}/\boldsymbol{U}_1)^e \cap T_1/U_1 = 1,$$

so that $\boldsymbol{F}/\boldsymbol{U}_1$ is a subdirect product of $\boldsymbol{F}/\boldsymbol{T}_1$ and $\boldsymbol{F}/A_1(\boldsymbol{F})^e$; or

$$\mathscr{U}_1 = \mathscr{T} \vee \mathfrak{A}_u \circ \mathfrak{A} \vee \mathscr{P}$$

where $\mathscr{P}$ has finite exponent, by (5.3.8). Consideration of non-nilpotent critical bigroups and the fact that $\mathfrak{A}_u \circ \mathfrak{A} \leqslant \mathscr{V}(d, s)$, gives that $u = 1$. By modularity we have that $\mathscr{U}_i = \mathscr{T} \vee (\mathscr{U}_i \wedge \mathscr{P})$ $(i \in I^+)$ and (4.2.29) finishes the proof that the chain (5.4.11) breaks off.

5.5. *Subvarieties of* $\mathfrak{A} \circ \mathfrak{A}$ *of infinite exponent*

In this final section of this chapter we bring together the threads of the other sections. Most of these have in part involved the proving of special cases of (5.5.1) below, necessary to prove minimum condition on $\Lambda(\mathfrak{A} \circ \mathfrak{A})$, a fact which we need to prove (5.5.1) in general. Theorems 5.5.1 and 5.3.1 complement one another and should be read in conjunction.

Theorem. *Let $\mathscr{B}$ be a proper subvariety of $\mathfrak{A} \circ \mathfrak{A}$ containing $\mathfrak{E} \circ \mathfrak{A}$. Then there exists a torsion-free subvariety $\mathscr{T}$ of $\mathscr{B}$ and a unique natural number u such that*

$$\mathscr{B} = \mathscr{T} \vee \mathfrak{A}_u \circ \mathfrak{A} \vee \mathscr{P},$$

where $\mathscr{P}$ has finite exponent. If $\mathscr{B} \nleqslant \mathfrak{A}_m \circ \mathfrak{A} \vee \mathfrak{A} \circ \mathfrak{A}_n$ for any m, n then $\mathscr{T}$ is unique; otherwise $\mathscr{T} \vee \mathfrak{E} \circ \mathfrak{A}$ is unique. (5.5.1)

Corollary. *The proper join-irreducible torsion-free subvarieties of $\mathfrak{A} \circ \mathfrak{A}$ are precisely*

$$\mathfrak{E} \circ \mathfrak{A}, \quad \mathfrak{E} \circ \mathfrak{A}_{p^\beta}, \quad \mathfrak{A} \circ \mathfrak{A}_s, \quad \mathscr{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A} \quad (\beta \geqslant 0, p \text{ prime}, c \geqslant 2, s \geqslant 1).$$

Moreover, every torsion-free subvariety $\mathscr{T}$ of $\mathfrak{A} \circ \mathfrak{A}$ can be written uniquely as a finite irredundant join of join-irreducible torsion-free subvarieties and this is the only way that $\mathscr{T}$ can be expressed as a finite irredundant join of join-irreducibles. (5.5.2)

The bivariety $\mathscr{P}$ in (5.5.1) is in general not unique. Indeed

Example. *The bivarieties $\mathscr{P}$ in the statements of* (5.5.1), (5.3.1), (5.3.8) *and the bivariety $\mathscr{N}$ in the statement of* (4.2.30) *are in general not unique, even when minimally chosen.* (5.5.3)

It can be seen from the lattice $\Lambda(\mathfrak{A}_4 \circ \mathfrak{A}_4 \wedge \mathscr{N}_3)$ drawn in § 4.4 that

$$(\mathfrak{A}_2 \circ \mathfrak{A}_4 \wedge \mathscr{N}_3) \vee (\mathfrak{A}_4 \circ \mathfrak{A}_4 \wedge \mathscr{N}_2) = (\mathfrak{A}_2 \circ \mathfrak{A}_4 \wedge \mathscr{N}_3) \vee (\mathfrak{A}_4 \circ \mathfrak{A}_2 \wedge \mathscr{N}_3), \tag{5.5.4}$$

and that, moreover, $\mathfrak{A}_4 \circ \mathfrak{A}_4 \wedge \mathscr{N}_2$ and $\mathfrak{A}_4 \circ \mathfrak{A}_2 \wedge \mathscr{N}_3$ are minimal with respect to preserving this join. Joining each side of (5.5.4) with $\mathfrak{A}_2 \circ \mathfrak{A}$ disposes of (5.5.1), (5.3.8), and with $\mathfrak{A}_2 \circ \mathfrak{A}_4$ of (4.2.30). Joining $\mathfrak{A} \circ \mathfrak{A}_2$ to each side of

$$(\mathfrak{A}_4 \circ \mathfrak{A}_2 \wedge \mathscr{N}_3) \vee (\mathfrak{A}_4 \circ \mathfrak{A}_4 \wedge \mathscr{N}_2) = (\mathfrak{A}_4 \circ \mathfrak{A}_2 \wedge \mathscr{N}_3) \vee (\mathfrak{A}_2 \circ \mathfrak{A}_4 \wedge \mathscr{N}_3)$$

disposes of (5.3.1).

Proof of (5.5.1). Let T be the torsion subgroup of $A = A_1(\boldsymbol{F}_\infty(\mathscr{B}))$. Now using (5.4.10) it follows as in (5.3.7) that T has finite exponent, and that

$$\mathscr{B} = \mathscr{T} \vee \mathscr{U},$$

where $\mathscr{T}$ is torsion-free and $\mathscr{U}$ is a subvariety of $\mathfrak{A}_m \circ \mathfrak{A}$ say. Theorem 5.3.8 completes the proof of existence.

Let $\mathscr{B}$ have a bilaw $[y_1, rz_1^s]^t$ (by virtue of (5.1.2)) where t is chosen minimally. Now $[y_1, rz_1^s]^t$ is a bilaw in $\mathfrak{A}_u \circ \mathfrak{A}$ and so u divides t. Also, for some suitably large natural number n, $\mathscr{B}$ has a bilaw $[y_1, rz_1^{sn}]^u$ and the minimality of t ensures that $t \leqslant u$ whence $u = t$, thus proving the uniqueness of u.

Next suppose that $\mathscr{B}$ has a decomposition

$$\mathscr{B} = \mathscr{T}' \vee \mathfrak{A}_u \circ \mathfrak{A} \vee \mathscr{P}'.$$

If q in $A_1(\boldsymbol{Q}_2)$ is a bilaw in $\mathscr{T}$, then q^μ is a bilaw in $\mathscr{B}$ for suitably chosen μ, and hence q is a bilaw in $\mathscr{T}'$. Now if $\mathscr{B}$ is not a subvariety of $\mathfrak{A}_m \circ \mathfrak{A} \vee \mathfrak{A} \circ \mathfrak{A}_n$ for any m, n then neither $\mathscr{T}$ nor $\mathscr{T}'$ has a bilaw z_1^ν $(\nu \geqslant 1)$ and therefore we have shown $\mathscr{T} \geqslant \mathscr{T}'$; similarly, $\mathscr{T} \leqslant \mathscr{T}'$ so that $\mathscr{T} = \mathscr{T}'$. In any case neither $\mathscr{T} \vee \mathfrak{E} \circ \mathfrak{A}$ nor $\mathscr{T}' \vee \mathfrak{E} \circ \mathfrak{A}$ has a bilaw z_1^ν $(\nu \geqslant 1)$, and both are torsion-free, so the proof just given shows that $\mathscr{T} \vee \mathfrak{E} \circ \mathfrak{A} = \mathscr{T}' \vee \mathfrak{E} \circ \mathfrak{A}$.

LEMMA. *Suppose that* $\mathscr{T}_0, \ldots, \mathscr{T}_\lambda$ *are torsion-free and chosen from the list in* (5.5.2), *such that*

$$\mathscr{T}_0 \leqslant \bigvee\{\mathscr{T}_i : 1 \leqslant i \leqslant \lambda\}.$$

Then for some $i \geqslant 1$, $\mathscr{T}_0 \leqslant \mathscr{T}_i$. (5.5.5)

The proof of (5.5.5) uses familiar arguments with critical bigroups and it is omitted.

LEMMA. *If* $\mathscr{T} = \bigvee\{\mathscr{B}_i : 1 \leqslant i \leqslant \mu\}$ *is torsion-free and* $\mathscr{T}_i$ *is the largest torsion-free subvariety of* $\mathscr{B}_i$, *then*

$$\mathscr{T} = \bigvee\{\mathscr{T}_i : 1 \leqslant i \leqslant \mu\}. \qquad (5.5.6)$$

Again the proof is routine and it is omitted.

Proof of (5.5.2). From (5.2.3) we see that the bivarieties displayed are the only candidates for join-irreducibility: and (5.5.5), (5.5.6) show that they are indeed join-irreducible. Moreover, if we write an arbitrary proper torsion-free subvariety of $\mathfrak{A} \circ \mathfrak{A}$ as an irredundant join of join-irreducibles (as (5.4.10) ensures we can) then (5.5.5) and (5.5.6) ensure the uniqueness we want.

Finally, bringing together the results of this section and (4.2.33), we have

THEOREM. *The non-nilpotent join-irreducible subvarieties of* $\mathfrak{A} \circ \mathfrak{A}$ *are*:

(*a*) *Infinite exponent*:

$$\mathfrak{A} \circ \mathfrak{A}_s, \quad \mathscr{N}_c(\mathfrak{E} \circ \mathfrak{A}_s) \wedge \mathfrak{A} \circ \mathfrak{A} \quad (s \geqslant 1, c \geqslant 2); \quad \mathfrak{A}_{p^\alpha} \circ \mathfrak{A} \quad (p \textit{ prime}, \alpha \geqslant 0);$$

and (*b*) *Finite exponent*:

(i) $\mathscr{L}(\mathfrak{E} \circ \mathfrak{A}_p) \wedge \mathfrak{A} \circ \mathfrak{A}$, *p prime*, $\mathscr{L}$ *join-irreducible of p-power exponent*, $\mathscr{L}\phi \neq \mathfrak{E} \circ \mathfrak{E}$;

(ii) $\mathscr{U}(\mathfrak{E} \circ \mathfrak{A}_t) \wedge \mathfrak{A} \circ \mathfrak{A}$, $\mathscr{U}$ *join-irreducible of p-power exponent* (*p prime*), $\mathscr{U}\phi \neq \mathfrak{E} \circ \mathfrak{E}$, $p \nmid t$, $t > 1$. (5.5.7).

Proof. It only needs verifying that $\mathfrak{A}_{p^\alpha} \circ \mathfrak{A}$ is join-irreducible and this follows easily from (5.3.8).

Chapter 6. Applications to varieties of metabelian groups

In § 6.1 we give a reduction, to the case of prime-power exponent, of the problem of determining all join-irreducibles in $\Lambda(\mathfrak{A}\mathfrak{A})$. Section 6.2 contains results relating to distributivity in $\Lambda(\mathfrak{A}\mathfrak{A})$. Section 6.3 contains classification results similar to (4.1.8) for the case m nearly prime to n, and for arbitrary m, n provided that the class of p-groups in the varieties concerned is suitably restricted.

6.1. *Join-irreducibles in* $\Lambda(\mathfrak{A}\mathfrak{A})$

As promised earlier we include statements of two results of Kovács & Newman. In appendix I the proofs of these are sketched. A *torsion-free* variety of groups is one generated by torsion-free groups.

Theorem (L. G. Kovács & M. F. Newman). *Let $\mathfrak{B}$ be a proper subvariety of $\mathfrak{A}\mathfrak{A}$. Then there exists a unique torsion-free variety $\mathfrak{T}$ and a unique natural number u such that*

$$\mathfrak{B} = \mathfrak{T} \vee \mathfrak{A}_u\mathfrak{A} \vee \mathfrak{P},$$

where $\mathfrak{P}$ has finite exponent. (6.1.1)

Theorem (L. G. Kovács & M. F. Newman). *The varieties of groups $\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}\mathfrak{A}$ $(c, s \geqslant 1)$ are torsion-free and join-irreducible. Every torsion-free subvariety of $\mathfrak{A}\mathfrak{A}$ can be uniquely expressed as an irredundant join of these torsion-free join-irreducibles.* (6.1.2)

Corollary. *The join-irreducibles in $\Lambda(\mathfrak{A}\mathfrak{A})$ which do not have finite exponent come from the list*

$$\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}\mathfrak{A}, \quad \mathfrak{A}_{p^\alpha}\mathfrak{A}, \quad p \text{ prime}, \quad \alpha > 0, \quad c, s \geqslant 1. \qquad (6.1.3)$$

It is here that chapter 4 becomes relevant, enabling a further reduction to be made. First we note some preliminaries, beginning with a converse to (3.3.4).

Theorem. *Let G be a non-nilpotent metabelian critical group. There is a complement B for G' in G, and (G, G', B) is a critical bigroup. Moreover, all such bigroups carried by G are isomorphic.* (6.1.4)

Proof. Since G is not nilpotent there is a natural number u such that

$$1 \neq G_{(u)} = G_{(u+1)} = \ldots .$$

Now $G_{(u)}$ is abelian and, by Schenkman (1955), is complemented in G, all such complements being conjugate. The same proof as that of (3.2.1) can now be used together with (3.1.2); the conjugacy of complements ensures that different (G, G', B) are isomorphic.

Lemma. *If q is a biword, then there exist words $w_1, \ldots, w_d$ such that q is a bilaw in the non-nilpotent critical bigroup $\mathbf{G}$ of $\mathfrak{A} \circ \mathfrak{A}$ if and only if $w_1, \ldots, w_d$ are laws in (the group) G. Conversely, if w is a word, then there exists a biword q' such that w is a law in the carrier of the bigroup $\mathbf{H}$ if and only if q' is a bilaw in $\mathbf{H}$.* (6.1.5)

Proof. We may assume the biword q written, modulo the bilaws of $\mathfrak{A} \circ \mathfrak{A}$, in one of the forms

$$y_1^\alpha, \quad z_1^\beta, \quad \prod_{i=1}^{t} [y_1, z_1^{\lambda_{i1}}, \ldots, z_r^{\lambda_{ir}}]^{\alpha_i}$$

by (2.2.3). The words

$$[x_1, x_2]^\alpha, \quad [x_1, x_2, x_3^\beta], \quad \prod_{i=1}^{t} [x_1, x_2, x_3^{\lambda_{i1}}, \ldots, x_{r+2}^{\lambda_{ir}}]^{\alpha_i},$$

respectively then do what we want. For the converse, $q' = w(y_1 z_1, \ldots, y_s z_s)$ will serve.

Let Λ_0 be the set of all subvarieties of $\mathfrak{A}\mathfrak{A}$ which are generated by non-nilpotent critical groups, and consider Λ_0 as a lattice under the inherited inclusion order. Similarly, define Λ_0' to be the set of sub-bivarieties of $\mathfrak{A}\circ\mathfrak{A}$ which are generated by non-nilpotent critical bigroups. By virtue of (3.3.4) and (6.1.4) there is a one-to-one correspondence between (isomorphism classes of) non-nilpotent critical groups in $\mathfrak{A}\mathfrak{A}$ and (isomorphism classes of) non-nilpotent critical bigroups in $\mathfrak{A}\circ\mathfrak{A}$. Using (6.1.5) we see that this one-to-one correspondence induces a lattice isomorphism θ: $\Lambda_0 \to \Lambda_0'$ defined by:

$$(\mathrm{var}\,\{G_i : G_i \text{ non-nilpotent, critical, } i\in I\})\,\theta = \mathrm{svar}\,\{\mathbf{G}_i : i\in I\}. \tag{6.1.6}$$

Lemma. *Suppose that* $\{G_i : i\in I\}$ *is a set of metabelian critical groups of bounded exponent which is critical-factor closed, and that* G *in* $\mathrm{var}\,\{G_i : i\in I\}$ *is critical. Then there exists a finite subset* I' *of* I *such that for* i *in* I', $|\sigma G_i| = |\sigma G|$, $|K_i| = |K|$ *and*

$$G\in \mathrm{var}\,\{G_i : i\in I'\}. \tag{6.1.7}$$

Proof. Use a result of Kovács & Newman (1966, 1.12) and one of Higman (1959, lemma 4.2).

Theorem. *Let* $\mathfrak{B}$ *be a metabelian variety of finite exponent. Then if* $\mathfrak{B}$ *does not have a prime-power exponent it is join-irreducible if and only if*

$$\mathfrak{B} = (\mathscr{S}(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A})\,\tau,$$

where $\mathscr{S}$ *is join-irreducible of* p*-power exponent with* $\mathfrak{E}\circ\mathfrak{E} \nleq \mathscr{S}\phi$, *and where* $t > 1$ *and* p *does not divide* t. (6.1.8)

Proof. If $\mathfrak{B}$ is join-irreducible then it must be generated by its non-nilpotent critical groups. Hence $\mathfrak{B}\theta$ is join-irreducible in Λ_0' and therefore, by virtue of (4.1.9), join-irreducible in $\Lambda(\mathfrak{A}\circ\mathfrak{A})$. From (4.1.16)

$$\mathfrak{B}\theta = \mathscr{S}(\mathfrak{E}\circ\mathfrak{A}_t)\wedge\mathfrak{A}\circ\mathfrak{A},$$

where $\mathscr{S}$ and t have the properties asserted. Now since $\mathfrak{B}$ is in Λ_0, $\mathfrak{B}$ is a subvariety of $\mathfrak{B}\theta\tau$; and by (1.7.2), $(\mathfrak{B}\theta)\,\tau \leqslant (\mathfrak{B}\sigma)\,\tau \leqslant \mathfrak{B}$ whence $\mathfrak{B} = \mathfrak{B}\theta\tau$.

Conversely, one verifies easily that for $\mathscr{B}$ in Λ_0', $\mathscr{B} = \mathscr{B}\tau\theta$, so that if $\mathscr{B}$ is join-irreducible and of finite exponent, $\mathscr{B}\tau = \mathscr{B}\theta^{-1}$ is join-irreducible in Λ_0 and therefore, by (6.1.7), in $\Lambda(\mathfrak{A}\mathfrak{A})$ also.

Thus the classification of non-nilpotent join-irreducibles in $\Lambda(\mathfrak{A}\mathfrak{A})$ is reduced to the case of prime-power exponent. Brooks (1968) has dealt with $\Lambda(\mathfrak{A}_p\mathfrak{A}_{p^2})$ obtaining a result similar to (4.2.30), and is working on the general case.

6.2. *Distributivity in* $\Lambda(\mathfrak{A}_m\mathfrak{A}_n)$

The results (4.1.11) and (4.1.13) have their analogues here: we adapt the notation in (4.1.7) in the obvious way.

Theorem. *For each prime* p *dividing* m, *the mapping* ν_p: $\Lambda(\mathfrak{B})\to\Lambda(\mathfrak{B}_p)$ *defined by*

$$\mathfrak{W}\nu_p = \mathfrak{W}\wedge\mathfrak{B}_p, \quad \mathfrak{W}\leqslant\mathfrak{B} = \mathfrak{A}_m\mathfrak{A}_n$$

is a lattice homomorphism. The ν_p *provide a subdirect decomposition of* $\Lambda(\mathfrak{B})$. (6.2.1)

Proof. Use (6.1.7).

Theorem. *Let* Δ_p *be the set of positive divisors of* n_p, *ordered by division. Then if* $|\Delta_p| = s_p$, $\Lambda(\mathfrak{B}_p)$ *is a sub-direct product of* Δ_p, $\Lambda(\mathfrak{U}_p)$ *and* $s_p - 1$ *copies of* $\Phi(\mathscr{U}_p)$. (6.2.2)

Proof. Write, for $\mathfrak{W} \leqslant \mathfrak{B}_p$,

$$\mathfrak{W}\xi_{p0} = \mathfrak{W} \wedge \mathfrak{A}_{n_p}, \quad \mathfrak{W}\xi_{p1} = \mathfrak{W} \wedge \mathfrak{U}_p,$$

$$\mathfrak{W}\xi_p = \text{var}\{G \in \mathfrak{W} : G \text{ critical, non-nilpotent}\},$$

and define $\xi_{pt} = \xi_p \theta \lambda_{pt}$ for $t|n_p$ (where λ_{pt} is defined in (4.1.12)). It follows from (6.1.7) that ξ_p, ξ_{pt} are lattice homomorphisms for $t \in \{0\} \cup \Delta_p$. Moreover, for $\mathfrak{W} \leqslant \mathfrak{B}_p$,

$$\mathfrak{W} = \mathfrak{W}\xi_{p0} \vee \mathfrak{W}\xi_{p1} \vee \mathfrak{W}\xi_p,$$

and therefore, using (4.1.8), $\mathfrak{W}$ is determined uniquely by the set $\{\mathfrak{W}\xi_{pt} : t \in \{0\} \cup \Delta_p\}$. This completes the proof.

COROLLARY. *If $\mathfrak{W} \leqslant \mathfrak{A}_m \mathfrak{A}_n$ then $\Lambda(\mathfrak{W})$ is distributive if and only if for each prime p dividing m and each t dividing n_p, $\Lambda(\mathfrak{W})\,\nu_p\,\xi_{pt}$ is distributive.* (6.2.3)

COROLLARY. *If m is nearly prime to n, then $\Lambda(\mathfrak{A}_m \mathfrak{A}_n)$ is distributive.* (6.2.4)

Proof. We need (6.2.3), (4.3.13) and (4.1.6) and Newman & Kovác's (1970) result that $\Lambda(\mathfrak{A}_{p^\alpha}\mathfrak{A}_p)$ is distributive (see appendix II).

COROLLARY. *Let $\mathfrak{W}$ be a variety of metabelian groups of bounded exponent in which p-groups have class at most p. Then $\Lambda(\mathfrak{W})$ is distributive. On the other hand, if $\mathfrak{X}$ is the subvariety of $\mathfrak{A}_{p^2}\mathfrak{A}_{p^2N}$ ($p \nmid N, N \neq 1$) which consists of groups whose Sylow p-subgroups have class at most $p+1$, then $\Lambda(\mathfrak{X})$ is not distributive.* (6.2.5)

Proof. Use (6.2.3), (4.4.4), (4.1.6) and (4.4.8) to reduce to the question of showing that p-power exponent subvarieties of class p form a distributive lattice. For class $p-1$ this follows from Weichsel (1967) or Brisley (1967); and for class p from Warren Brisley's unpublished description of $\Lambda(\mathfrak{A}\mathfrak{A} \wedge \mathfrak{B}_{p^\alpha} \wedge \mathcal{N}_p)$ (see appendix II).

6.3. *Classification results*

Ideally one would like to have, for varieties of metabelian groups of finite exponent, a result similar to (4.1.8), classifying subvarieties of $\mathfrak{A}_m \mathfrak{A}_n$ in terms of those of prime-power exponent. Regrettably (4.1.8) does not generalize as one would hope. However we can give rather elaborate conditions under which (4.1.8) does have an analogue: two classification results follow as corollaries. The notation of (4.1.7) and §§ 1.7 and 6.2 is used.

Let $\mathfrak{A}_n \leqslant \mathfrak{W} \leqslant \mathfrak{A}_m \mathfrak{A}_n$ and define

$$\mathfrak{W}^\dagger = \vee\{\mathfrak{W}_{pt}\mathfrak{A}_t \wedge \mathfrak{A}_{p^{\alpha(p,t)}}\mathfrak{A}_n : p|m, t|n_p\}, \tag{6.3.1}$$

where for each $p|m$:

$$\mathfrak{W}_{p1} = \mathfrak{W}\nu_p,$$
$$\mathfrak{W}_{pt} = \mathfrak{W}\nu_p\,\xi_{pt}\,\tau, \quad 1 \neq t|n_p,$$
$$p^{\alpha(p,1)} \| m,$$
$$p^{\alpha(p,t)} = \max \exp\{G' : G \in \mathfrak{W} \text{ critical}, \exp \sigma G = p, |K| = t\}.$$

Notice that $\mathfrak{W} \leqslant \mathfrak{W}^\dagger$, and for $p|m$:

$$\left.\begin{array}{ll} \text{(i)} & \mathfrak{W}_{pt} \leqslant \mathfrak{U}_p, \text{ and for } t > 1,\ \mathfrak{W}_{pt}\sigma \in \Phi(\mathscr{U}_p), \\ \text{(ii)} & t|u|n_p \text{ implies } \mathfrak{W}_{pu} \leqslant \mathfrak{W}_{pt},\ \alpha(p,u) \leqslant \alpha(p,t). \\ \text{(iii)} & p^{\alpha(p,t)} \leqslant \min(p^{\alpha(p,1)}, \exp \mathfrak{W}_{pt}). \end{array}\right\} \tag{6.3.2}$$

THEOREM. *If $\mathfrak{A}_n \leqslant \mathfrak{W} \leqslant \mathfrak{A}_m \mathfrak{A}_n$, then $\mathfrak{W} = \mathfrak{W}^\dagger$ if and only if for $p|m$ and $1 \neq t|n_p$, $\mathfrak{W}\nu_p\,\xi_{pt}$ is open.* (6.3.3)

Proof. It is clear that $\mathfrak{W}$, $\mathfrak{W}^{\dagger}$ have the same nilpotent critical groups. Suppose for some p and some $t(\neq 1)$ $\mathfrak{W}_{pt}$ is non-trivial, and let $\boldsymbol{P}$ in $\mathfrak{W}_{pt}\sigma$ be a critical bigroup. By (3.4.7) there exists critical $\boldsymbol{G}$ with $\boldsymbol{F}^* \cong \boldsymbol{P}$ and $|K| = t$. Now $\mathfrak{W} = \mathfrak{W}^{\dagger}$ if and only if for all such choices of p and $\boldsymbol{P}$, G belongs to $\mathfrak{W}$; that is if and only if $G \in \operatorname{var}\{G_i : i \in I,\ G_i \text{ critical}, \exp \sigma G_i = p,\ |K_i| = t,\ G_i \in \mathfrak{W}\}$ (by (6.1.7)), which is true if and only if $\boldsymbol{G} \in \operatorname{svar}\{\boldsymbol{G}_i : i \in I\}$ (by (6.1.4), (6.1.5)), or $\boldsymbol{F}^* \in \mathfrak{W}\nu_p\,\xi_{pt}$. Hence $\mathfrak{W} = \mathfrak{W}^{\dagger}$ if and only if for all such choices of $\boldsymbol{P}$ in $\mathfrak{W}_{pt}\sigma$, $\boldsymbol{P}$ belongs to $\mathfrak{W}\nu_p\,\xi_{pt}$; in other words, if and only if $\mathfrak{W}\nu_p\,\xi_{pt}$ are all open.

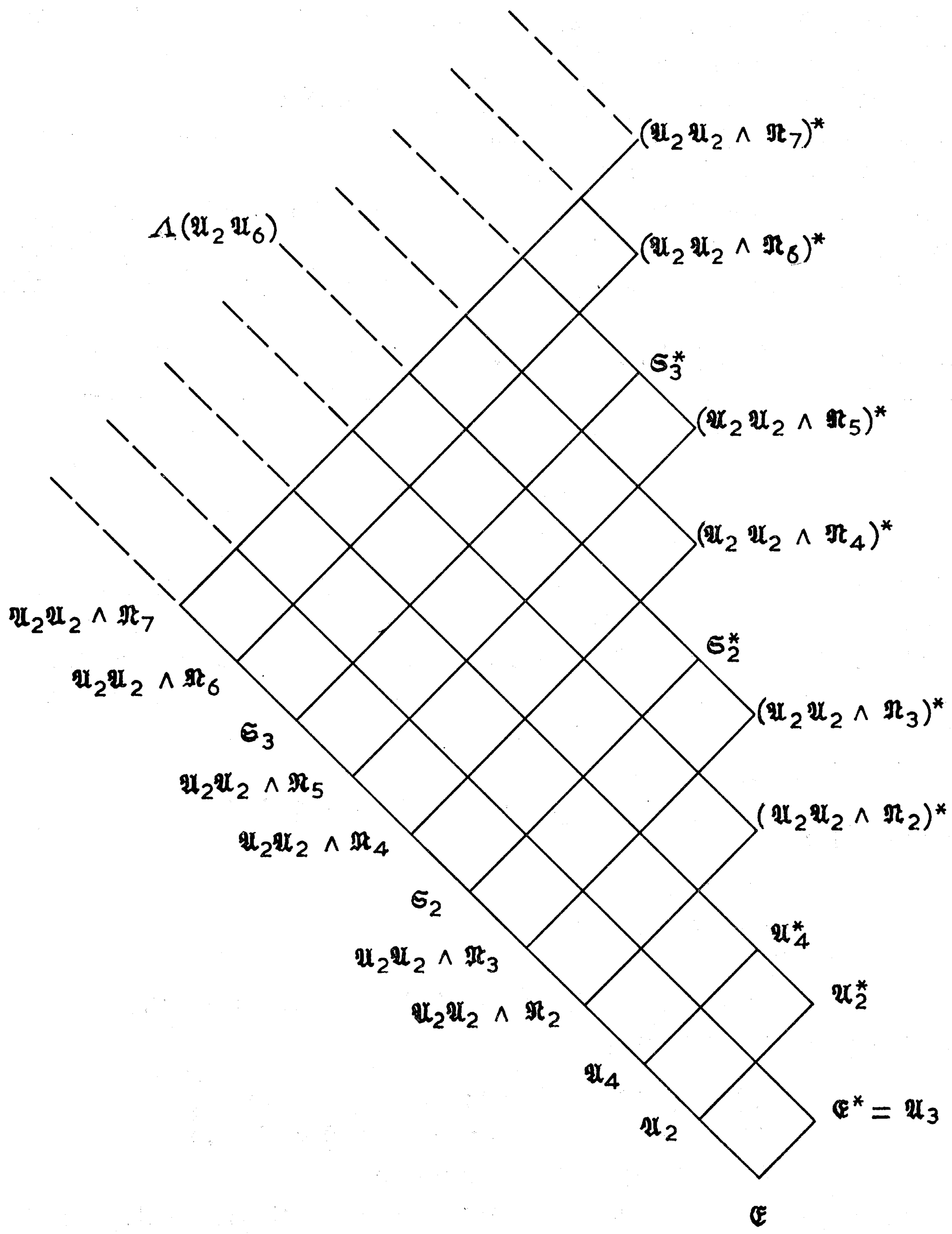

FIGURE 2

COROLLARY. $(\mathfrak{W}^{\dagger})^{\dagger} = \mathfrak{W}^{\dagger}$. (6.3.4)

The decomposition (6.3.1) for $\mathfrak{W}^{\dagger}$ is essentially unique, as the following theorem shows.

THEOREM. *If* $\mathfrak{W}'_{pt}$ $(p|m, t|n_p)$ *are subvarieties of* $\mathfrak{A}_m\mathfrak{A}_n$ *satisfying* (6.3.1) *and* (6.3.2) *and* $\alpha'(p,t)$ *are natural numbers satisfying* (6.3.1) *and* (6.3.2) *then for* $t > 1$ *and* $p|m$,

$$\mathfrak{W}'_{pt}\sigma = \mathfrak{W}_{pt}\sigma (= \mathfrak{W}\nu_p\,\xi_{pt}) \quad \text{and} \quad \alpha'(p,t) = \alpha(p,t). \tag{6.3.5}$$

Proof. The first assertion follows in the usual way from (3.4.7) and (6.1.7). The equality of $\alpha(p,t)$, $\alpha'(p,t)$ follows because there exist in $\mathfrak{W}^{\dagger}$ the critical A-groups of Cossey (1966, theorem 4.2.2) with derived group exponents $p^{\alpha(p,t)}$, $p^{\alpha'(p,t)}$ and derived factor groups of order t.

It remains to reduce the hypotheses of (6.3.3) so as to get some positive results of existence and uniqueness for certain metabelian varieties along the lines of (4.1.8). The two theorems following achieve this; their proofs follow from (4.3.11) and (4.4.4) respectively. (The unpublished work of Kovács & Newman and of Brisley (see appendix II), together with (4.3.11) and (4.4.4) then provides a complete classification in these cases).

THEOREM. *If m is nearly prime to n then* $\mathfrak{W}\nu_p\,\xi_{pt}$ *is open for each* $\mathfrak{W} \leqslant \mathfrak{A}_m\mathfrak{A}_n$, $p|m$, $1 \neq t|n_p$. (6.3.6)

THEOREM. *If* $\mathfrak{X}$ *is a variety of metabelian groups of bounded exponent e in which p-groups have class at most p, then* $\mathfrak{W}\nu_p\,\xi_{pt}$ *is open for each* $\mathfrak{W} \leqslant \mathfrak{X}$, $p|e$, $1 \neq t|e_p$. (6.3.7)

By way of illustration, the lattice $\Lambda(\mathfrak{A}_2\mathfrak{A}_6)$ is drawn, using the Kovács & Newman description of $\Lambda(\mathfrak{A}_2\mathfrak{A}_2)$.‡ Notice that even in this simplest case the $\mathfrak{W}_{pt}$ need not be unique (cf. example 1.7.11); both the varieties $\mathfrak{N}_{2\lambda-1} \wedge \mathfrak{A}\mathfrak{A}$ and $\mathfrak{S}_\lambda$ give rise to the same non-nilpotent critical groups; alternatively

$$(\mathfrak{N}_{2\lambda-1} \wedge \mathfrak{A}\mathfrak{A})\,\sigma = \mathfrak{S}_\lambda\,\sigma.$$

Finally, we give an example of a sub-bivariety of $\mathfrak{A}_2 \circ \mathfrak{A}_4$ which is not open, thereby showing that $\mathfrak{W} < \mathfrak{W}^{\dagger}$ can happen.

EXAMPLE. Consider the bigroups $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ carried by the groups

$$C_2 \operatorname{wr} (C_4 \times C_2), \quad C_2 \operatorname{wr} (C_4 \times C_2 \times C_2)/(C_2 \operatorname{wr} (C_4 \times C_2 \times C_2))_{(6)},$$
$$C_2 \operatorname{wr} (C_4 \times C_4)/(C_2 \operatorname{wr} (C_4 \times C_4))_{(6)}$$

respectively. It is tedious though not difficult to verify that F_1, F_2, F_3 generate the same variety of groups (indeed the carrier of any bigroup in $\mathfrak{A}_2 \circ \mathfrak{A}_4$ which does not satisfy $[y_1, 3z_1, z_2]$, and has class 5, generates $\mathfrak{A}_2\mathfrak{A}_4 \wedge \mathfrak{N}_5$). However $\boldsymbol{F}_1$ has the bilaws

$$[y_1, 2z_1, 2z_2], \quad [y_1, 2z_1, z_2, z_3]\,[y_1, z_1, 2z_2, z_3]\,[y_1, z_1, z_2, 2z_3],$$

$\boldsymbol{F}_2$ has the first, but not the second, and $\boldsymbol{F}_3$ has neither.

In fact if G is the critical group with $\boldsymbol{F}^* \cong \boldsymbol{F}_1$ and $|K| = 3$, say, while $\mathfrak{V}_1 = \operatorname{var} F_1 = \operatorname{var} \mathfrak{A}_3(G)$, $\mathfrak{V}_2 = \operatorname{var} \mathfrak{A}_3\mathfrak{A}_2(G)$ and $\mathfrak{V}_3 = \operatorname{var} \mathfrak{A}_3\mathfrak{A}_2\mathfrak{A}_2(G)$ then it is clear from the above example that var G is at best second maximal in

$$\mathfrak{V}_1\mathfrak{A}_3 \wedge \mathfrak{V}_2\mathfrak{A}_6 \wedge \mathfrak{V}_3\mathfrak{A}_{12}.$$

Much of the work presented here is contained in Bryce (1967), and was done while I held a Commonwealth Postgraduate Award which was supplemented by the Australian National University. My supervisor was Dr L. G. Kovács to whom my grateful thanks are due, both for his supervision and, more recently, for the ready access he has allowed me to unpublished work of his own.

‡ See appendix II. The variety $\mathfrak{S}_\lambda$ here is just $\mathfrak{N}_{\lambda 2}$.

Appendix I. Proofs of (6.1.1) and (6.1.2)

In a forthcoming paper ('Varieties of metabelian groups', submitted to the *Bulletin of the Australian Mathematical Society*) Kovács & Newman announce the results (6.1.1) and (6.1.2). Since my chapter 5 relied so heavily on their methods, they have kindly allowed me to publish complete proofs here. The sketch to follow, together with their announcement, provides a proof of (6.1.1) and (6.1.2).

(*a*) (cf. (5.1.1)). If m, t are coprime, $C_m \operatorname{wr} C_t$ generate $\mathfrak{A}_m \mathfrak{A}_t$. $C_m \operatorname{wr} C_\infty$ generates $\mathfrak{A}_m \mathfrak{A}$. (P. J. Cossey (1966) and Hanna Neumann (1967, 17.6 and 22.44).)

(*b*) (cf. (5.1.2), similar proof). If $\mathfrak{A}_m \mathfrak{A} \nleq \mathfrak{U} \leqslant \mathfrak{A}\mathfrak{A}$, $\mathfrak{U}$ has a law $[x, y, rz^s]^t$, $m \nmid t$.

(*c*) (cf. (5.1.4), similar proof). If $\mathfrak{U} < \mathfrak{A}\mathfrak{A}$, $\mathfrak{U}$ has a law $[x, y, z_1^s, \ldots, z_c^s]^t$.

(*d*) (cf. Higman (1955). If H is a finitely generated, torsion-free, nilpotent group and Π an infinite set of primes then H is residually of prime ($\in \Pi$) exponent.

(*e*) For $m, n \in I^+$, $\mathfrak{A}_m \mathfrak{A} \wedge \mathfrak{A}\mathfrak{A}_n = \mathfrak{A}_m \mathfrak{A}_n \vee \mathfrak{A}$.

Proof. Let $F = F_\infty(\mathfrak{A}_m \mathfrak{A} \wedge \mathfrak{A}\mathfrak{A}_n)$. Now $\mathfrak{A}_n(F)/F'$ is free abelian; hence, since $\mathfrak{A}_n(F)$ is abelian, F' is complemented in $\mathfrak{A}_n(F)$. Therefore $\mathfrak{A}_m \mathfrak{A}_n(F) \cap F' = \mathfrak{A}_m(F') = 1$, whence the result.

(*f*) (cf. (5.3.8)). If $\mathfrak{U} \leqslant \mathfrak{A}_m \mathfrak{A}$ then there exists a unique $u | m$ such that $\mathfrak{U} = \mathfrak{A}_u \mathfrak{A} \vee \mathfrak{P}$ where $\mathfrak{P}$ has finite exponent.

Proof. As in proof of (5.3.8) there exists $u | m$ and $N \in I^+$ such that $\mathfrak{A}_u \mathfrak{A} \leqslant \mathfrak{U}$, and $[x, y^N]^u$ is a law in $\mathfrak{U}$. From (*e*) if $F = F_\infty(\mathfrak{A}_m \mathfrak{A})$, $\{[x, y^N]\}(F) \geqslant \mathfrak{A}_m \mathfrak{A}_N(F) \cap F'$ so that

$$\{[x, y^N]^u\}(F) \geqslant \mathfrak{A}_u(\mathfrak{A}_m \mathfrak{A}_N(F) \cap F') = \mathfrak{A}_m \mathfrak{A}_N(F) \cap \mathfrak{A}_u(F')$$

since $\mathfrak{A}_m \mathfrak{A}_N(F) \cap F'$ is complemented in F' (by M. S. Brooks (1968, 1.2.2)). Therefore $\mathfrak{A}_u \mathfrak{A} \leqslant \mathfrak{U} \leqslant \mathfrak{A}_u \mathfrak{A} \vee \mathfrak{A}_m \mathfrak{A}_N$ and modularity gives the result.

Proof of (6.1.2). Similar to that of (5.2.3): use (*c*) in place of (5.1.4), (*d*) in place of (5.1.10), (6.3.3) and (6.3.6) in place of (4.1.8). The torsion-freeness of $\mathfrak{N}_c \mathfrak{A}_s \wedge \mathfrak{A}^2$ is proved in Kovács & Newman's announcement.

Proof of (6.1.1). Similar to that of (5.5.1), appealing to Cohen (1967) instead of to (5.4.10), and to (*f*) instead of to (5.3.8).

Appendix II

For convenience we quote here two unpublished theorems which have been referred to several times in the text.

(*a*) Theorem (Kovács & Newman; cf. (4.3.11)). *The lattice* $\Lambda(\mathfrak{A}_{p^\alpha}\mathfrak{A}_p)$ *is distributive. Every subvariety* $\mathfrak{U}$ *of* $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ *can be written*

$$\mathfrak{U} = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}_{p^\tau} \wedge \bigwedge\{\mathfrak{A}_{p^\beta}\mathfrak{N}_{\nu_\beta} : 0 \leqslant \beta \leqslant \alpha-1\}$$

for some $\tau \in \{0, \ldots, \alpha-1\}$ *and* ν_β *from the set* $\{1, 2, \ldots, p-1, p^*, p, \ldots, \lambda p-1, \lambda p^*, \lambda p, \ldots, \omega\}$. *There is a condition that may be imposed on the* ν_β, *similar to* (4.3.11) (ii), *but much more complicated, which makes them unique. The variety* $\mathfrak{N}_{\lambda p^*}$ *is defined by the law* $\prod_{i=2}^{\lambda p}[x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\lambda p}]$.

(*b*) Theorem (Warren Brisley; cf. (4.4.4)). *Each verbal subgroup V of the free group F of rank* ∞ *in the variety* $\mathfrak{A}\mathfrak{A} \wedge \mathfrak{B}_{p^\alpha} \wedge \mathfrak{N}_p$ *can be written uniquely as*

$$V = F_{(1)}^{p^{\alpha_1}} F_{(2)}^{p^{\alpha_2}} \ldots F_{(p)}^{p^{\alpha_p}} F_{(p^*)}^{p^{\alpha_{p+1}}},$$

where $\alpha \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_{p+1} \geqslant 0$, $\alpha_p - \alpha_{p+1} \leqslant 1$, $\alpha_{p+1} \leqslant \alpha - 1$; *and where* $F_{(p^*)}$ *is defined by the word* $\prod_{i=2}^{p}[x_i, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p]$.

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